Mathematical Morphology on Complete Lattices for Imperfect Information Processing
Application to Image Understanding and Spatial Reasoning

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Lattices and information processing

Lattices: core mathematical structure in many information processing problems.

Examples:

- soft computing (fuzzy sets, bipolar information),
- knowledge representation,
- logics,
- formal concept analysis,
- automated reasoning,
- decision making,
- image processing and understanding,
- information retrieval,
- etc.

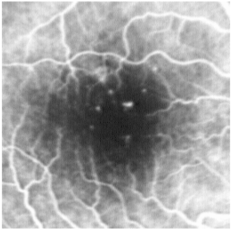
Mathematical morphology on complete lattices.

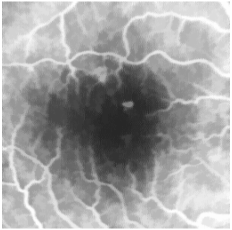
Mathematical Morphology for Spatial Information

Matheron (mid-1960's), Serra (1982)

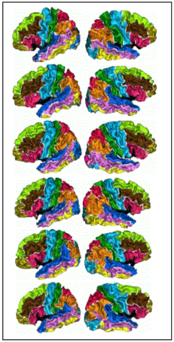
- A theory of space.
- Widely used in image processing and interpretation.
- At different levels (local, regional, structural...).
- For different tasks (filtering, enhancement, segmentation, interpretation, spatial knowledge modeling...).

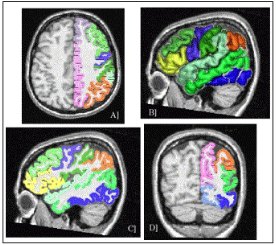
Filtering

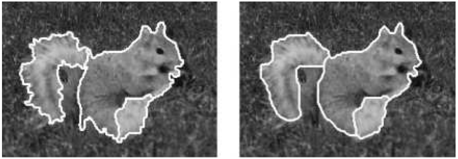




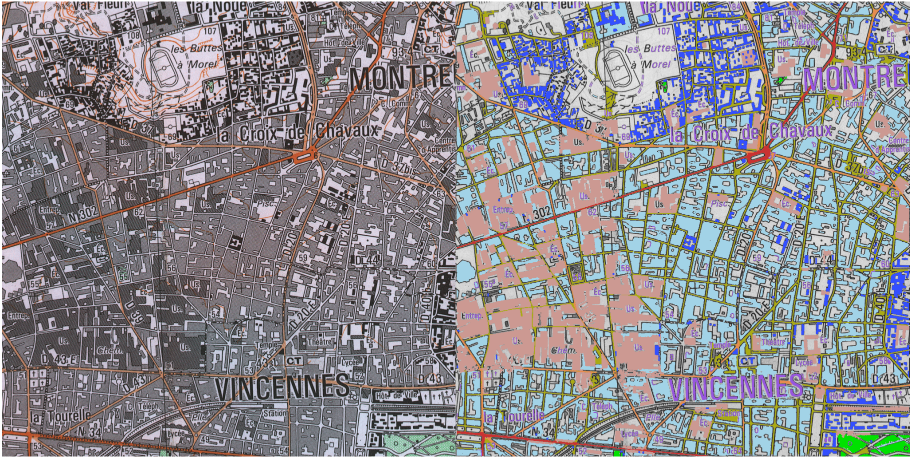
Segmentation



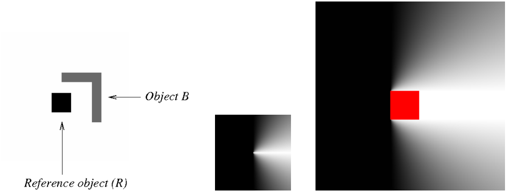




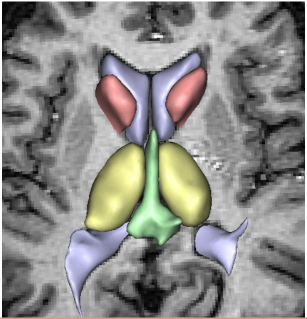
Interpretation



Knowledge modeling What is the region *to the right of R*? Is *B to the right of R (and to which degree)*?



Spatial reasoning



Formal framework: complete lattices

- Lattice: $(\mathcal{T}, \leq)$ ($\leq$ partial ordering) such that $\forall (x, y) \in \mathcal{T}, \exists x \vee y$ and $\exists x \wedge y$.
- Complete lattice: every family of elements (finite or not) has a smallest upper bound and a largest lower bound.
- $\Rightarrow$ contains a smallest element 0 and a largest element I:

$$0 = \bigwedge \mathcal{T} = \bigvee \emptyset \text{ and } I = \bigvee \mathcal{T} = \bigwedge \emptyset$$

- Examples of complete lattices:
  - $(\mathcal{P}(E), \subseteq)$: complete lattice, Boolean (complemented and distributive)
  - functions of $\mathbb{R}^n$ in $\overline{\mathbb{R}}$ for the partial ordering $\leq$: $f \leq g \Leftrightarrow \forall x \in \mathbb{R}^n, \ \ f(x) \leq g(x)$
  - partitions
  - logics (propositional logics, modal logics...)
  - fuzzy sets, bipolar fuzzy sets
  - rough sets and fuzzy rough sets
  - formal concepts
  - ...

Algebraic dilations and erosions

Heijmans, Ronse (1990)

Complete lattices $(\mathcal{T}, \leq)$, $(\mathcal{T}', \leq')$

Algebraic dilation: $\delta : \mathcal{T} \rightarrow \mathcal{T}'$ such that

$$\forall (x_i) \in \mathcal{T},\ \delta(\vee_i x_i) = \vee'_i \delta(x_i)$$

Algebraic erosion: $\varepsilon : \mathcal{T}' \rightarrow \mathcal{T}$ such that

$$\forall (x_i) \in \mathcal{T}',\ \varepsilon(\wedge'_i x_i) = \wedge_i \varepsilon(x_i)$$

Properties:

- $\delta(0) = 0'$ (in $\mathcal{P}(E), 0 = \emptyset$)
- $\varepsilon(I') = I$ (in $\mathcal{P}(E), I = E$)
- δ increasing, ε increasing
- in $\mathcal{P}(\mathbb{R}^n)$, $\delta(X) = \cup_{x \in X} \delta(\{x\})$

Adjunctions

$\delta : \mathcal{T} \to \mathcal{T}'$, $\varepsilon : \mathcal{T}' \to \mathcal{T}$, (ε, δ) adjunction if:

$$\forall x \in \mathcal{T}, \forall y \in \mathcal{T}',\ \delta(x) \leq' y \Leftrightarrow x \leq \varepsilon(y)$$

Properties:

- $\delta(0) = 0'$ and $\varepsilon(I') = I$
- (ε, δ) adjunction $\Rightarrow \varepsilon =$ algebraic erosion and $\delta =$ algebraic dilation
- δ increasing $=$ algebraic dilation iff $\exists \varepsilon$ such that (ε, δ) is an adjunction $\Rightarrow \varepsilon =$ algebraic erosion and $\varepsilon(x) = \bigvee\{y \in \mathcal{T},\ \delta(y) \leq' x\}$
- ε increasing $=$ algebraic erosion iff $\exists \delta$ such that (ε, δ) is an adjunction $\Rightarrow \delta =$ algebraic dilation and $\delta(x) = \bigwedge\{y \in \mathcal{T}',\ \varepsilon(y) \geq x\}$
- $\varepsilon\delta \geq Id$ and $\delta\varepsilon \leq Id'$
- $\varepsilon\delta\varepsilon = \varepsilon$ and $\delta\varepsilon\delta = \delta$
- $\varepsilon\delta\varepsilon\delta = \varepsilon\delta$ and $\delta\varepsilon\delta\varepsilon = \delta\varepsilon$
- δ and ε increasing such that $\delta\varepsilon \leq Id'$ and $\varepsilon\delta \geq Id \Rightarrow (\varepsilon, \delta)$ adjunction

Morphological dilations and erosions

- On the lattice of the subsets of $\mathbb{R}^n$ or $\mathbb{Z}^n$, with inclusion:

$$\delta(X) = \cup_{x \in X} \delta(\{x\})$$

- $+$ invariance under translation
  $\Rightarrow \exists B,\ \delta(X) = D(X, B) = \{x, \check{B}_x \cap X \neq \emptyset\}$ (with $B_x = x + B$).
- $B =$ structuring element (neighborhood, binary relation).
- Same result on the lattice of functions.
- Similar results for erosion: $\exists B,\ \varepsilon(X) = E(X, B) = \{x, B_x \subseteq X\}$.

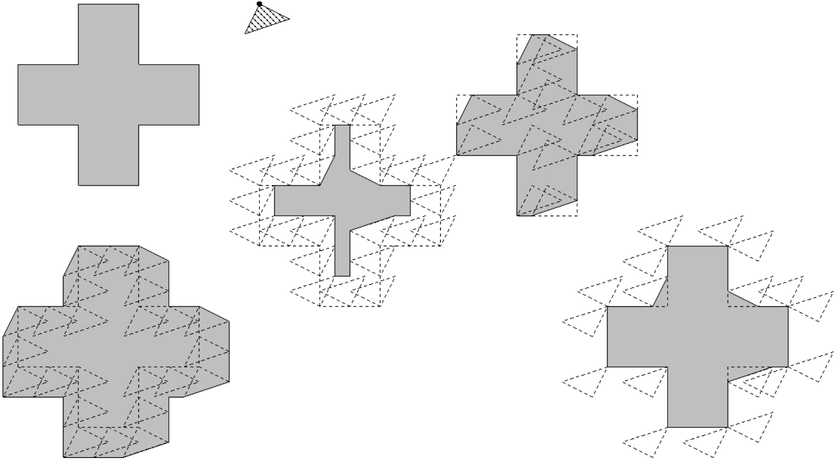
Derived operators: opening, closing, conditional (geodesic) operations, gradient...

Relaxing the assumption on invariance under translation: structuring elements varying in space (ex: projective geometry, omnidirectional images...).

Algebraic opening and closing

- Algebraic opening: γ increasing, idempotent and anti-extensive.
- Algebraic closing: φ increasing, idempotent and extensive.
- Examples: $\gamma = \delta\varepsilon$ and $\varphi = \varepsilon\delta$ with (ε, δ) adjunction.
- Invariance domain: $Inv(\varphi) = \{x \in \mathcal{T},\ \varphi(x) = x\}$.
- γ opening $\Rightarrow \gamma(x) = \bigvee\{y \in Inv(\gamma),\ y \leq x\}$.
- φ closing $\Rightarrow \varphi(x) = \bigwedge\{y \in Inv(\varphi),\ x \leq y\}$.
- (γ_i) openings $\Rightarrow \bigvee_i \gamma_i$ opening.
- (φ_i) closings $\Rightarrow \bigwedge_i \varphi_i$ closing.
- γ_1 and γ_2 openings $\Rightarrow$ equivalence between:
  1. $\gamma_1 \leq \gamma_2$
  2. $\gamma_1\gamma_2 = \gamma_2\gamma_1 = \gamma_1$
  3. $Inv(\gamma_1) \subseteq Inv(\gamma_2)$
- Similar result on closings.

A simple example



(Illustration: C. Ronse)

Lattice of fuzzy sets and fuzzy morphology

- Space $\mathcal{S}$ (e.g. $\mathbb{Z}^n$ or $\mathbb{R}^n$)
- $\mathcal{F}$: set of fuzzy sets on $\mathcal{S}$ – $\mu \in \mathcal{F}$, $\mu : \mathcal{S} \rightarrow [0, 1]$.
- Partial ordering:

$$\forall (\mu_1, \mu_2) \in \mathcal{F}^2, \mu_1 \leq \mu_2 \Leftrightarrow \forall x \in \mathcal{S}, \mu_1(x) \leq \mu_2(x)$$

- $(\mathcal{F}, \leq)$ = complete lattice
- $\wedge = \min$
- $\vee = \max$
- Algebraic dilation and erosion: as in any complete lattice

Morphological operations in the fuzzy case

Operators: t-norm t, t-conorm T, complementation c, implication I derived from T and c, residual implication I_R derived from t.

Fuzzy dilation of μ by ν:

$$\delta_\nu(\mu)(x) = \sup_{y \in \mathcal{S}} t[\nu(x-y), \mu(y)]$$

Fuzzy erosion of μ by ν:

- by duality:

$$\varepsilon_\nu(\mu)(x) = \inf_{y \in \mathcal{S}} T[c(\nu(y-x)), \mu(y)] = \inf_{y \in \mathcal{S}} I[\nu(y-x), \mu(y)]$$

- by adjunction:

$$\varepsilon_\nu(\mu)(x) = \inf_{y \in \mathcal{S}} I_R[\nu(y-x), \mu(y)]$$

Equivalence for Lukasiewicz operators (up to a bijective permutation on $[0,1]$).

Properties: as in classical morphology.

Structural information: spatial relations

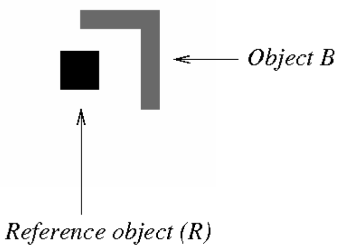
Expression of several spatial relations in terms of morphological operators:

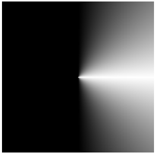
- adjacency
- distance (nearest point distance, Hausdorff distance)
- relative direction
- more complex relations (between, along...)

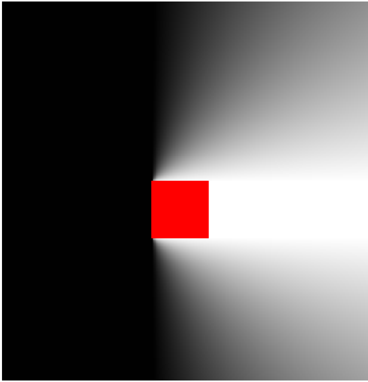
Two classes of relations:

- well defined in the crisp case
- vague even if objects are well defined

Example of directional relation







Minimum distance density

Binary discrete case:

$$d_N(X,Y) = n \Leftrightarrow \delta^n(X) \cap Y \neq \emptyset \text{ and } \delta^{n-1}(X) \cap Y = \emptyset$$

$$d_N(X,Y) = 0 \Leftrightarrow X \cap Y \neq \emptyset$$

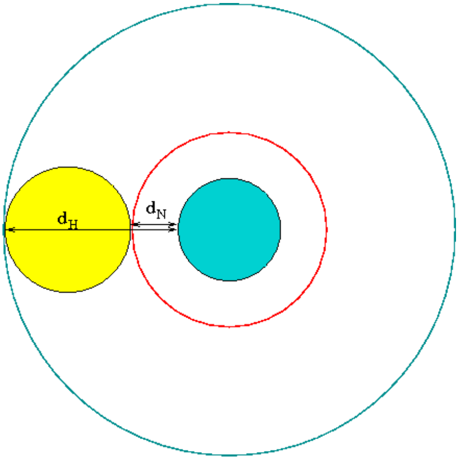
Degree to which the distance between μ and μ' is equal to n:

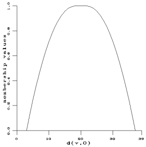
$$d_N(\mu,\mu')(n) = t[\sup_{x \in \mathcal{S}} t[\mu'(x), \delta_\nu^n(\mu)(x)], c[\sup_{x \in \mathcal{S}} t[\mu'(x), \delta_\nu^{n-1}(\mu)(x)]]]$$

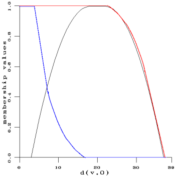
$$d_N(\mu,\mu')(0) = \sup_{x \in \mathcal{S}} t[\mu(x), \mu'(x)]$$

Hausdorff distance: similar equations.

Fuzzy distance: example




membership values
0.0
0.2
0.4
0.6
0.8
1.0
0
10
20
30
39
d(v,O)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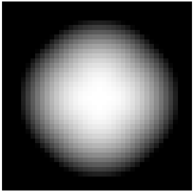
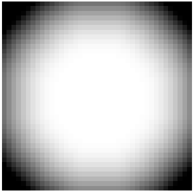
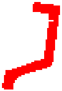
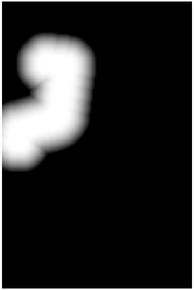
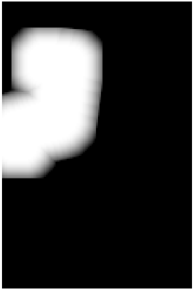
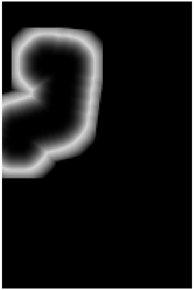




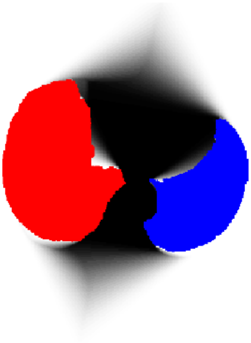


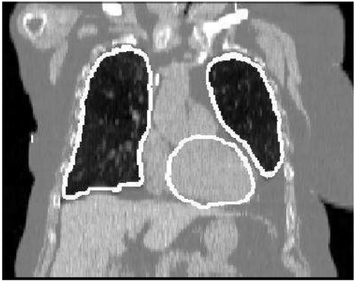


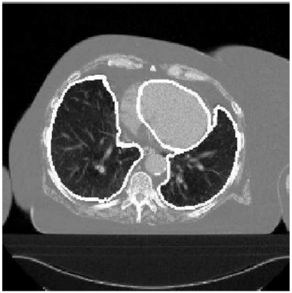




The heart is *between* the lungs







Morpho-Logics

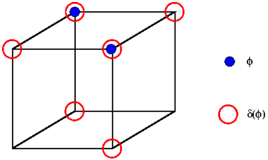
- Propositional logics and modal logics, associated complete lattice
- Dilations and erosions:

$$[\![\delta_B(\varphi)]\!] = \{\omega \in \Omega \mid \check{B}_\omega \wedge \varphi \textit{ consistent}\} \quad [\![\varepsilon_B(\varphi)]\!] = \{\omega \in \Omega \mid B_\omega \models \varphi\}$$

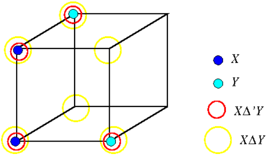
- Applications: revision, fusion, abduction, mediation, spatial reasoning (joint work with J. Lang, R. Pino-Perez, C. Uzcategui)
- Morphological expression of the max-fusion operator:

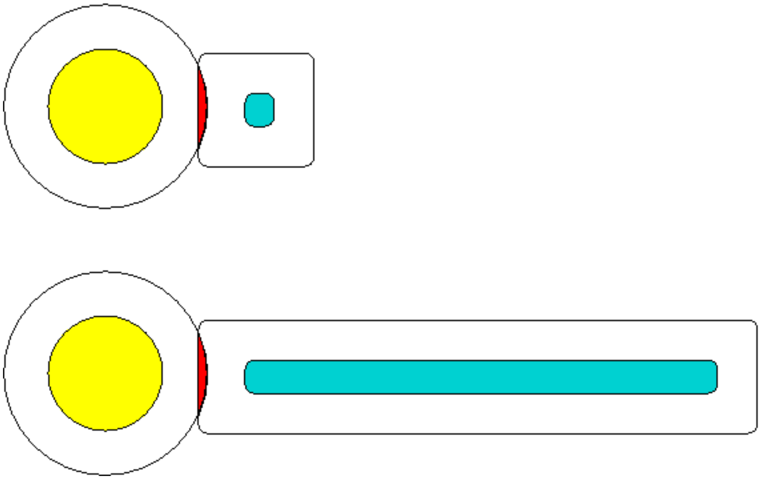
$$X \Delta Y = \delta_n(X) \cap \delta_n(Y) \quad \text{with } n = \min\{k : \delta_k(X) \cap \delta_k(Y) \neq \emptyset\}$$

Dilation



Fusion

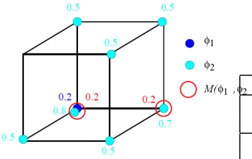




Extension to the fuzzy case

$[\![\varphi]\!]$ as a fuzzy set.
Example: Median set ($\mu_i = [\![\varphi_i]\!]$):

$$M(\mu_1, \mu_2) = \sup_{\lambda} t[\delta_{\lambda\nu}(\mu_1 \cap \mu_2), \varepsilon_{\lambda\nu}(\mu_1 \cup \mu_2)]$$



Models	abc	$\neg abc$	$a\neg bc$	$ab\neg c$	$\neg a\neg bc$	$\neg ab\neg c$	$a\neg b\neg c$	$\neg a\neg b\neg c$
φ_1	0	0	0	0	0	0.2	0	0
φ_2	0	0.5	0.5	0.5	0.5	0.8	0.5	0.7
$\delta_1(\varphi_1)$	0	0.2	0	0.2	0	0.2	0	0.2
$\varepsilon_1(\varphi_2)$	0	0	0	0	0.5	0.5	0.5	0.5
$\delta_2(\varphi_1)$	0.2	0.2	0	0.2	0.2	0.2	0.2	0.2
$\varepsilon_2(\varphi_2)$	0	0	0	0	0	0	0	0.5
$M(\varphi_1, \varphi_2)$	0	0	0	0	0	0.2	0	0.2

Other logics

Modal logics

Accessibility relation	Structuring element
$\square$	ε
$\diamond$	δ

Description logics

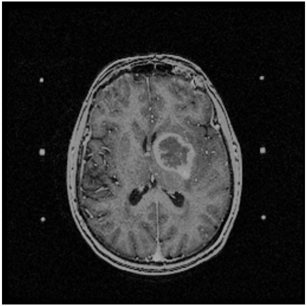
δ and ε as binary predicates.

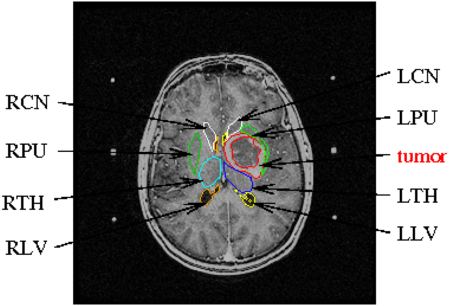
Information and bipolarity

- Positive information vs. negative information.
- Consistency: no overlap.
- No duality.
- (Links with interval-valued fuzzy sets and intuitionistic fuzzy sets.)
- Recent work (Dubois, Prade, et al.): fuzzy and possibilistic formalism.
- Important in the spatial domain:
  - image thresholding and edge detection (Chaira et al., Couto et al., Vlachos et al.)
  - spatial representations for classification (Charlier et al., Malek)
  - mathematical morphology (Bloch, Melange et al.)

Frameworks and examples

- Sets P and N with $P \cap N = \emptyset$.
- Fuzzy sets μ and ν in $\mathcal{S}$, with $\forall x \in \mathcal{S}, \mu(x) + \nu(x) \leq 1$ (e.g. degrees of preferences or constraints).
- Logical formulas φ and ψ with $\varphi \wedge \psi \models \perp$, and the models $[\![\varphi]\!]$ and $[\![\psi]\!]$ are sets or fuzzy sets.
- Utility functions, capacities, possibility distributions...





Agent 1:

- prefers to travel in Spain: $\varphi_1 = \mathit{Spain}$,
- has to stay in Europe: $\psi_1 = \neg(\mathit{Belgium} \vee \mathit{France} \vee \mathit{Spain} \vee \mathit{Portugal} \vee \mathit{Italy} \vee \mathit{Germany} \vee \mathit{TheNetherlands} \vee \ldots\}$.

Agent 2:

- prefers to travel in Morocco: $\varphi_2 = \mathit{Morocco}$,
- has to stay in a Mediterranean country: $\psi_2 = \neg(\mathit{Morocco} \vee \mathit{Spain} \vee \mathit{Italy} \vee \mathit{Portugal} \vee \ldots)$.

$\Rightarrow$ conflict!

- Extending preferences using dilation:

$$\delta(\varphi_1) = \mathit{Spain} \vee \mathit{France} \vee \mathit{Portugal} \vee \mathit{Morocco}$$

$$\delta(\varphi_2) = \mathit{Morocco} \vee \mathit{Algeria} \vee \mathit{Portugal} \vee \mathit{Spain}$$

- Introducing the constraints in order to satisfy the consistency requirements:

$$\varphi_1' = \delta(\varphi_1) \wedge \neg\psi_1 = \mathit{Spain} \vee \mathit{France} \vee \mathit{Portugal}$$

$$\varphi_2' = \delta(\varphi_2) \wedge \neg\psi_2 = \delta(\varphi_2)$$

- Fusion of preferences and constraints: conjunction of the preferences and disjunction of the constraints

$$(\varphi, \psi) = (\varphi_1' \wedge \varphi_2', \psi_1 \vee \psi_2) = (\mathit{Spain} \vee \mathit{Portugal}, \neg(\bigvee \mathit{Medit.\ and\ Eur.\ countries})$$

$\Rightarrow$ Solution for travelling in the set of models of these formulas.

Bipolar fuzzy sets

Modeling bipolarity and imprecision

Definition:

$$\mathcal{L} = \{(a, b) \in [0, 1]^2 \mid a + b \leq 1\}$$

$$(\mu, \nu) : \mathcal{S} \to [0, 1] \times [0, 1]$$

$$\forall x \in \mathcal{S}, \mu(x) + \nu(x) \leq 1$$

- μ: membership function (positive information)
- ν: non-membership function (negative information)
- do not necessarily come from the same source of information

(not the same semantics as interval-valued fuzzy sets)

Complete lattice structure

Symmetrical role of positive and negative information:

- Pareto partial ordering: $(a_1, b_1) \preceq_P (a_2, b_2)$ iff $a_1 \leq a_2$ and $b_1 \geq b_2$
- $(\mathcal{L}, \preceq_P)$ and $(\mathcal{B}, \preceq_P)$ = complete lattices
- Standard negation: (ν, μ)

Giving priority to the negative information:

- Lexicographic ordering $\preceq_{lex}$ (total ordering)
- $(\mathcal{L}, \preceq_{lex})$ and $(\mathcal{B}, \preceq_{lex})$ = complete lattices
- negation: reversing the order

- Supremum $(\bigvee_P, \bigvee_{lex})$, infimum $(\bigwedge_P, \bigwedge_{lex})$
- Smallest element: $(0, 1)$, largest element: $(1, 0)$

Bipolar fuzzy mathematical morphology

Algebraic definitions: dilation = commutes with the supremum, erosion = commutes with the infinum.

Using structuring elements:

- I: bipolar implication, C bipolar t-norm.
- Erosion as a bipolar degree of inclusion:

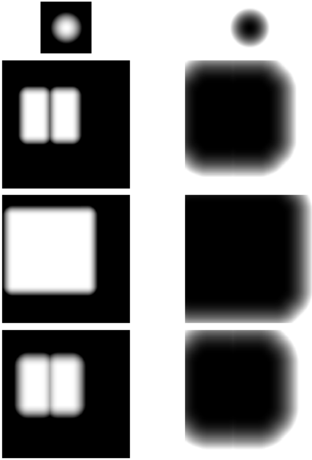
$$\varepsilon_{(\mu_B,\nu_B)}((\mu,\nu))(x) = \bigwedge_{y\in\mathcal{S}} I((\mu_B(y-x),\nu_B(y-x)),(\mu(y),\nu(y)))$$

- Dilation as a bipolar degree of intersection:

$$\delta_{(\mu_B,\nu_B)}((\mu,\nu))(x) = \bigvee_{y\in\mathcal{S}} C((\mu_B(x-y),\nu_B(x-y)),(\mu(y),\nu(y)))$$

Illustrative example

Positive information – Negative information



Bipolar fuzzy structuring element

Bipolar fuzzy set

Dilation using lexicographic min

Dilation using Pareto min

FCA: Adjunction and Galois connection

Equivalent concepts by reversing the order on one space.

$$\delta : A \to B, \varepsilon : B \to A$$
$$\delta(a) \leq_B b \Leftrightarrow a \leq_A \varepsilon(b)$$

increasing operators

$$\varepsilon\delta\varepsilon = \varepsilon, \delta\varepsilon\delta = \delta$$

$\varepsilon\delta =$ closing, $\delta\varepsilon =$ opening

$Inv(\varepsilon\delta) = \varepsilon(B)$, $Inv(\delta\varepsilon) = \delta(A)$

$\varepsilon(B) =$ Moore family

$\delta(A) =$ dual Moore family

$\delta =$ dilation: $\delta(\vee_A a_i) = \vee_B(\delta(a_i))$

$\varepsilon =$ erosion: $\varepsilon(\wedge_B b_i) = \wedge_A(\varepsilon(b_i))$

$$\alpha : B \to A, \beta : A \to B$$
$$a \leq_A \alpha(b) \Leftrightarrow b \leq_B \beta(a)$$

($\Leftrightarrow \beta(a) \leq'_B b$ with $\leq'_B \equiv \geq_B$)

decreasing operators

$$\alpha\beta\alpha = \alpha, \beta\alpha\beta = \beta$$

$\alpha\beta$ and $\beta\alpha =$ closings

$Inv(\alpha\beta) = \alpha(B)$, $Inv(\beta\alpha) = \beta(A)$

$\alpha(B)$ and $\beta(A) =$ Moore families

$\alpha(\vee_B b_i) = \wedge_A \alpha(b_i)$

$\beta(\vee_A a_i) = \wedge_B \beta(a_i)$ (anti-dilation)

Fuzzy extension

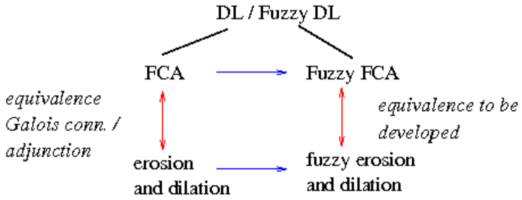
Belohlavek (1999): fuzzy Galois connection

$$A^{\uparrow}(y) = \bigwedge_x (A(x) \to I(x,y)), \quad B^{\downarrow}(x) = \bigwedge_y (B(y) \to I(x,y))$$

$\Rightarrow$ equivalent to a fuzzy anti-dilation

Dubois et al. (2007): possibilistic view

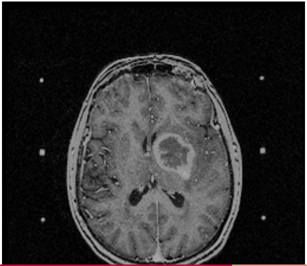
$$X^{\Pi} = \{y \mid \exists x \in X, I(x,y)\}, \quad X^{N} = \{y \mid \forall x, I(x,y) \Rightarrow x \in X\}$$
$$X^{\Delta} = \{y \mid \forall x \in X, I(x,y)\}, \quad X^{\nabla} = \{y \mid \exists x \in \bar{X}, \neg I(x,y)\}$$

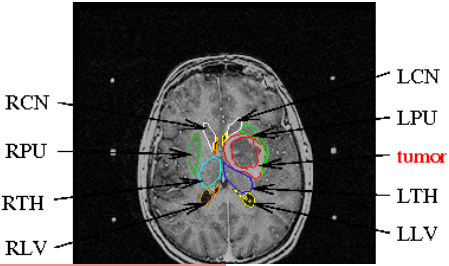


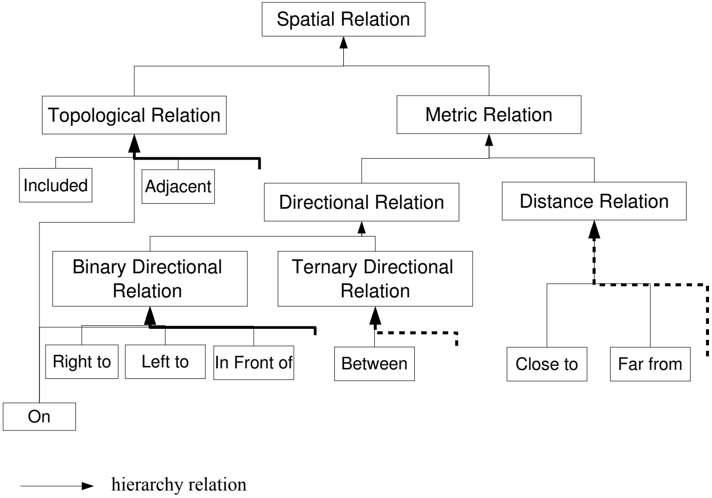
Fuzzy spatial relations and spatial reasoning

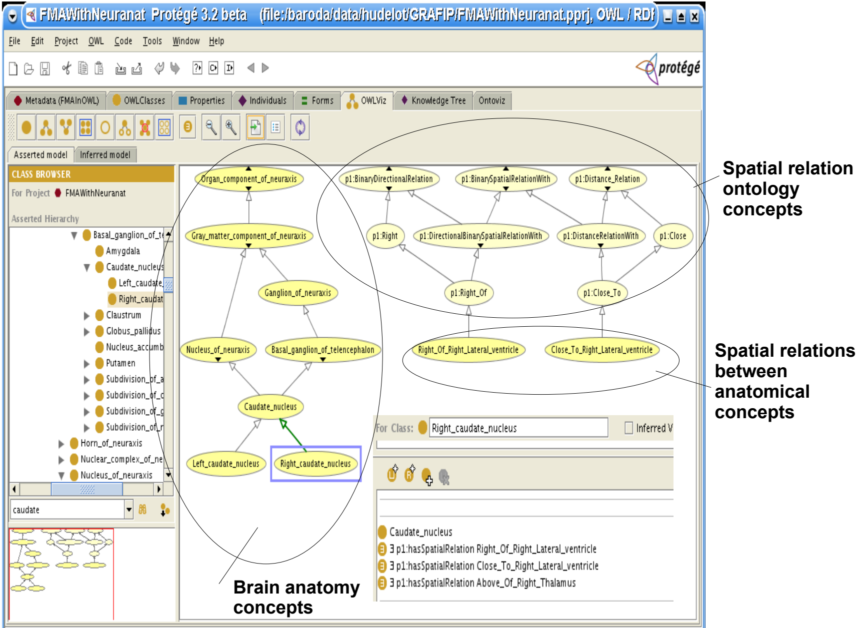
Example: brain imaging

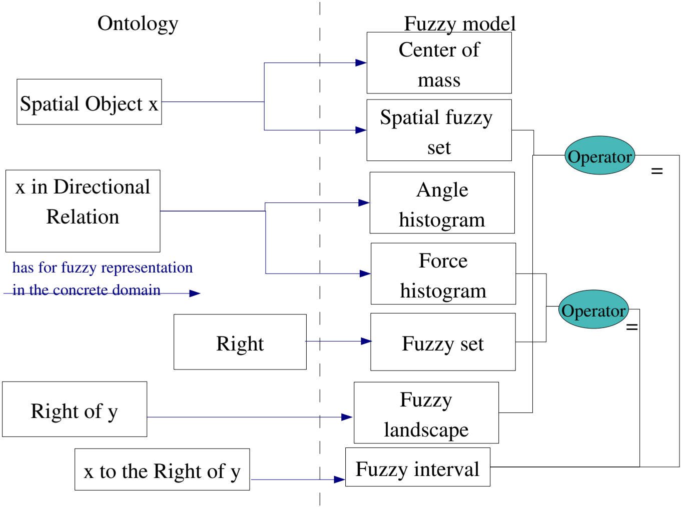
- Linguistic descriptions
  - direction: *the thalamus is below the lateral ventricle*
  - distance: *the lateral ventricles are far from the brain surface*
  - adjacency: *the thalamus is adjacent to the third ventricle*
  - symmetry: *homologous structures in both hemispheres*
- Fuzzy representations
- Attributed hierarchical graph (Colliot et al.) and ontologies (Hudelot et al.)



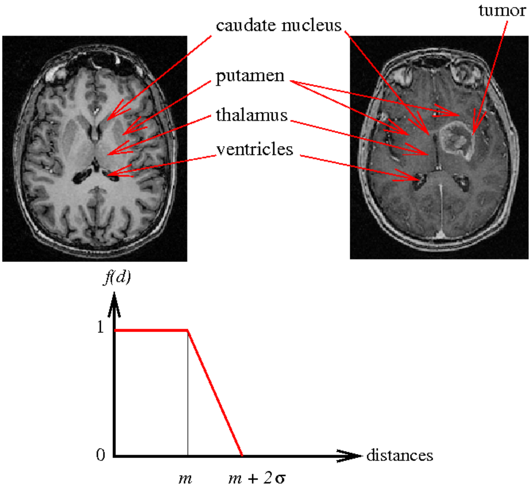


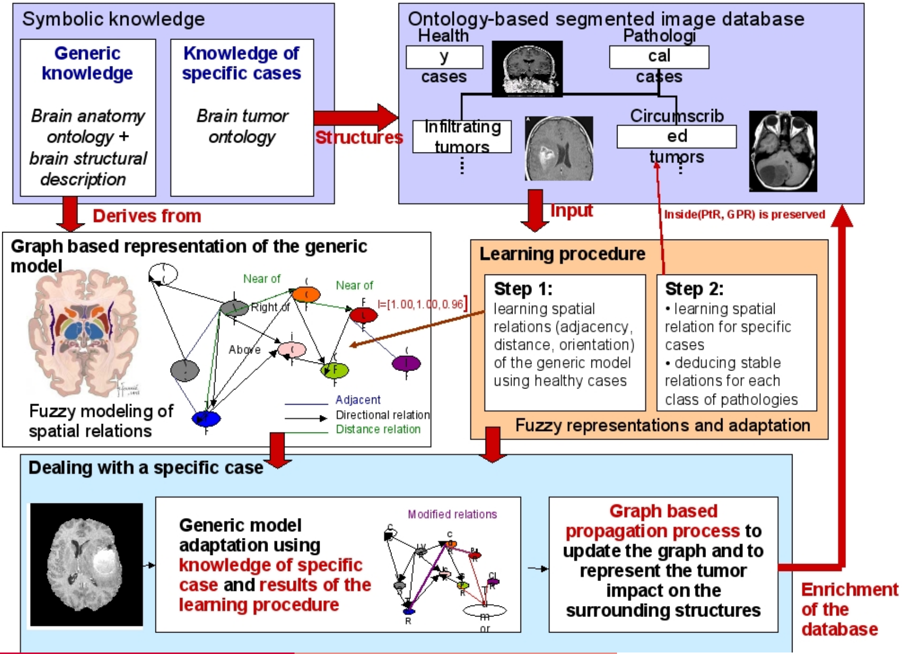

Spatial Relation
Topological Relation
Metric Relation
Included
Adjacent
Directional Relation
Distance Relation
Binary Directional Relation
Ternary Directional Relation
Right to
Left to
In Front of
Between
Close to
Far from
On
hierarchy relation


FMAWithNeuranat Protégé 3.2 beta (file:/baroda/data/hudelot/GRAFIP/FMAWithNeuranat.pprj, OWL / RDF)
Spatial relation ontology concepts
Spatial relations between anatomical concepts
Brain anatomy concepts
Organ_component_of_neuraxis
Gray_matter_component_of_neuraxis
Ganglion_of_neuraxis
Nucleus_of_neuraxis
Basal_ganglion_of_telencephalon
Caudate_nucleus
Left_caudate_nucleus
Right_caudate_nucleus
p1:BinaryDirectionalRelation
p1:BinarySpatialRelationWith
p1:Distance_Relation
p1:Right
p1:DirectionalBinarySpatialRelationWith
p1:DistanceRelationWith
p1:Close
p1:Right_Of
p1:Close_To
Right_Of_Right_Lateral_ventricle
Close_To_Right_Lateral_ventricle
For Class: Right_caudate_nucleus
Caudate_nucleus
∃ p1:hasSpatialRelation Right_Of_Right_Lateral_ventricle
∃ p1:hasSpatialRelation Close_To_Right_Lateral_ventricle
∃ p1:hasSpatialRelation Above_Of_Right_Thalamus

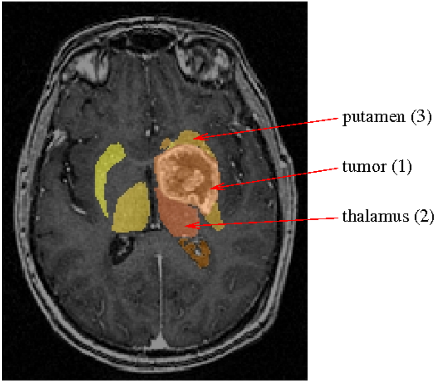

Ontology
Fuzzy model
Spatial Object x
Center of mass
Spatial fuzzy set
x in Directional Relation
Angle histogram
Force histogram
has for fuzzy representation in the concrete domain
Right
Fuzzy set
Right of y
Fuzzy landscape
x to the Right of y
Fuzzy interval
Operator
=
Operator
=

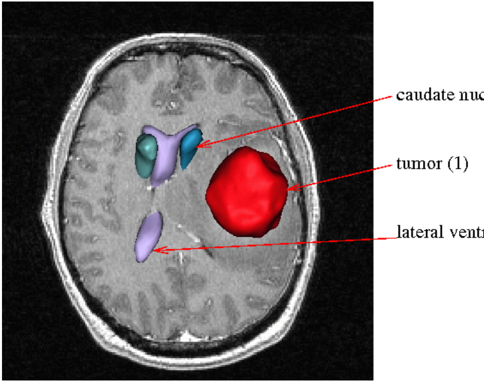
Learning spatial relations




Symbolic knowledge
Generic knowledge
Brain anatomy ontology + brain structural description
Knowledge of specific cases
Brain tumor ontology
Structures
Ontology-based segmented image database
Healthy cases
Pathological cases
Infiltrating tumors
Circumscribed tumors
Derives from
Input
Inside(PtR, GPR) is preserved
Graph based representation of the generic model
Near of
Near of
Right of
Above
I=[1.00,1.00,0.96]
Fuzzy modeling of spatial relations
Adjacent
Directional relation
Distance relation
Learning procedure
Step 1:
learning spatial relations (adjacency, distance, orientation) of the generic model using healthy cases
Step 2:
• learning spatial relation for specific cases
• deducing stable relations for each class of pathologies
Fuzzy representations and adaptation
Dealing with a specific case
Generic model adaptation using knowledge of specific case and results of the learning procedure
Modified relations
Graph based propagation process to update the graph and to represent the tumor impact on the surrounding structures
Enrichment of the database

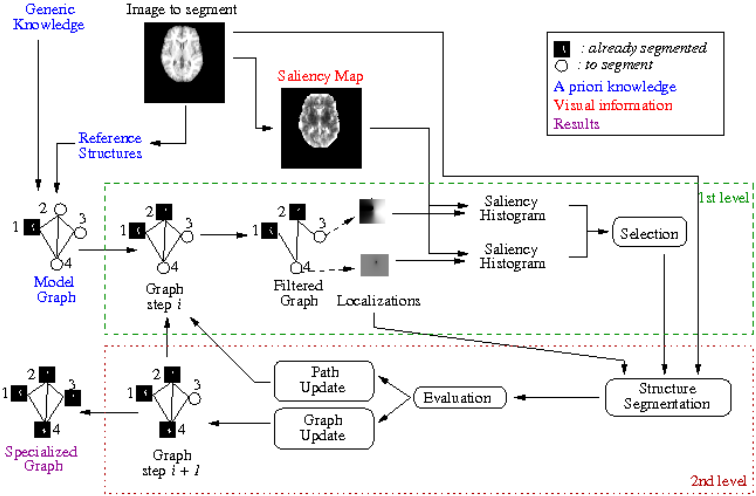
Khotanlou et al., Atif et al.





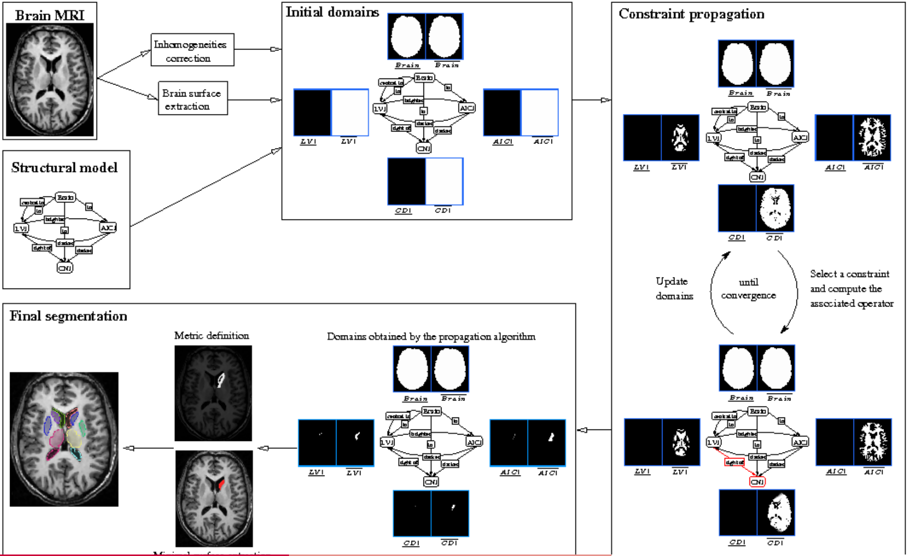
Optimizing the segmentation path

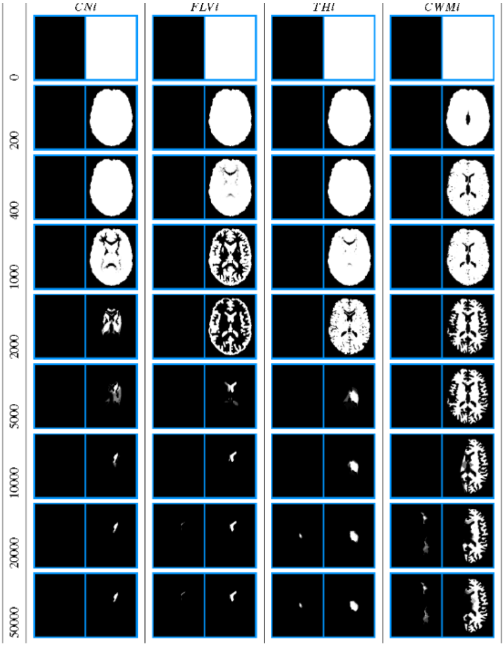
Reasoning in the graph and fusion with saliency information (Fouquier et al.)

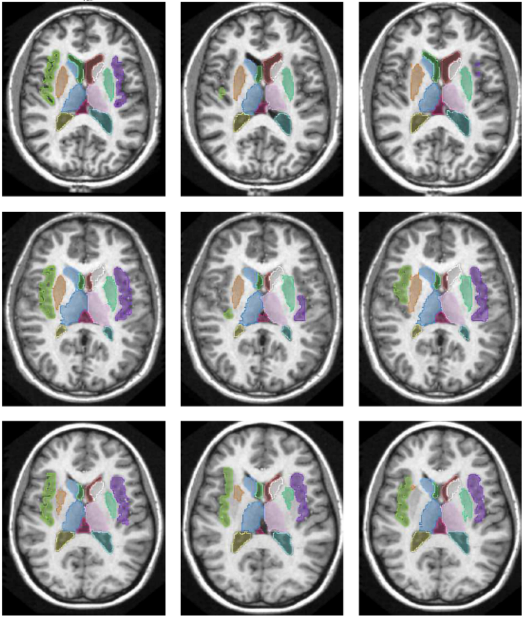


Global approach using a constraint network

Nempont et al.

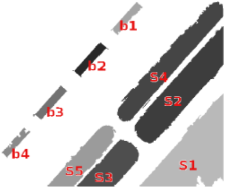



CNI
FLVI
THI
CWMI
0
200
400
1000
2000
5000
10000
20000
50000

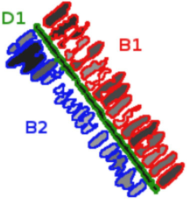


Remote sensing image understanding

- High resolution satellite image understanding.
- Collaboration with the CNES (PhD of Carolina Vanegas).
- Contributions:
  - modeling new spatial relations (surround, parallel, across, aligned...),
  - conceptual graphs integrating these relations,
  - new fuzzy CSP to deal with fuzzy complex relations and groups of objects,
  - understanding guided by conceptual graphs using FCSP.



A	B	$\mu_{\|\|N}(A,B)$	$\mu_{\|\|N}(B,A)$
b2	S4	0.94	0.55
b3	S5	0.97	0.87
b4	S5	0.89	0.66
S2	S4	0.97	0.97
S4	S1	0.87	0.94
S5	S3	0.90	0.95
S3	S1	0.78	0.43
b1	S4	0.90	0.69



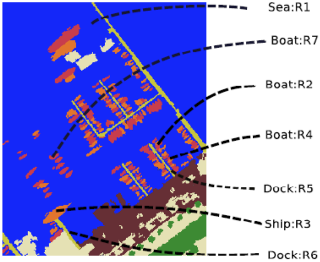
A	B	$\mu_{\|\|N}(A,B)$	$\mu_{\|\|N}(B,A)$
B1	D1	0.94	0.94
B2	D1	0.95	0.95
B1	B2	0.85	0.87

(a)

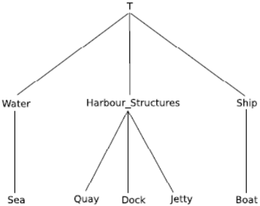
(b)

(c)

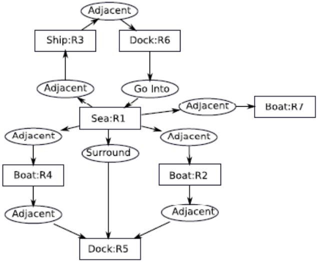
(a) Example image.



(b) Labeled image: The blue regions represent the sea, the red and orange represent ships or boats and the yellow regions represent the docks.



(c) Concept hierarchy T_C in the context of harbors.



(d) Conceptual graph representing the spatial organization of some elements of Figure 5.8(b).

Using modal logics

Examples (with $\Box \equiv \varepsilon$ and $\Diamond \equiv \delta$):

- tangential part: $\varphi \rightarrow \psi$ and $\Diamond\varphi \wedge \neg\psi$ consistent, or $\varphi \rightarrow \psi$ and $\varphi \wedge \neg\Box\psi$ consistent
- non tangential part: $\Diamond\varphi \rightarrow \psi$, or, $\varphi \rightarrow \Box\psi$
- external connection: $\varphi \wedge \psi$ inconsistent and $\Diamond\varphi \wedge \psi$ consistent (or $\varphi \wedge \Diamond\psi$ consistent)
- tangential proper part: tangential part and $\neg\varphi \wedge \psi$ consistent $(TPP(X, Y) = P(X, Y) \wedge \neg P(Y, X) \wedge \neg P(\delta(X), Y))$

Bipolarity and spatial reasoning

Directional information: the RPU is exterior (left on the image) of the union of RLV and RTH (positive information) and cannot be interior (negative information).
Distance information: the RPU is quite close to the union of RLV and RTH (positive information) and cannot be very far (negative information).

- Semantics of left (resp. right): fuzzy structuring element ν_L (resp.ν_R).

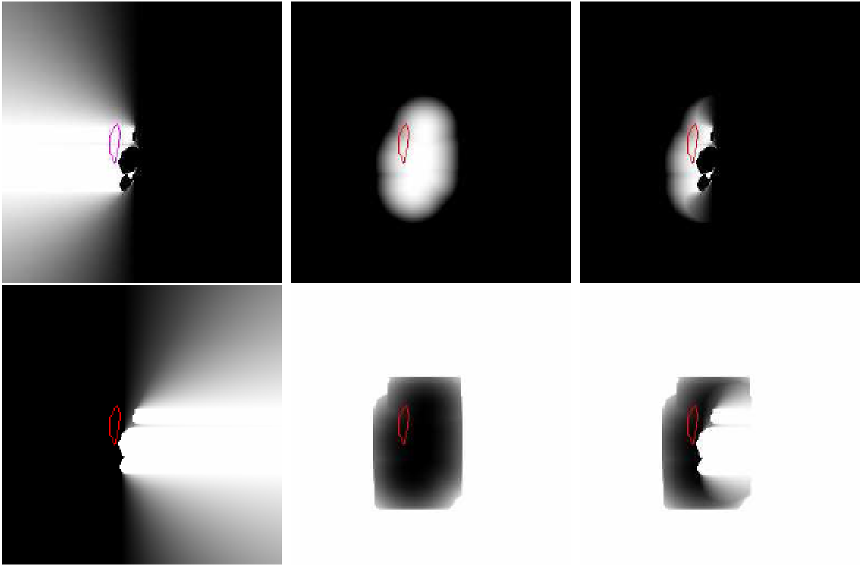
$$(\mu_{dir}, \nu_{dir}) = (\delta_{\nu_L}(\mathrm{RLV} \cup \mathrm{RTH}), \delta_{\nu_R}(\mathrm{RLV} \cup \mathrm{RTH}))$$

- Semantics of close (resp. far): ν_C (resp. ν_F).

$$(\mu_{dist}, \nu_{dist}) = (\delta_{\nu_C}(\mathrm{RLV} \cup \mathrm{RTH}), 1 - \delta_{1-\nu_F}(\mathrm{RLV} \cup \mathrm{RTH}))$$

- Conjunctive fusion:

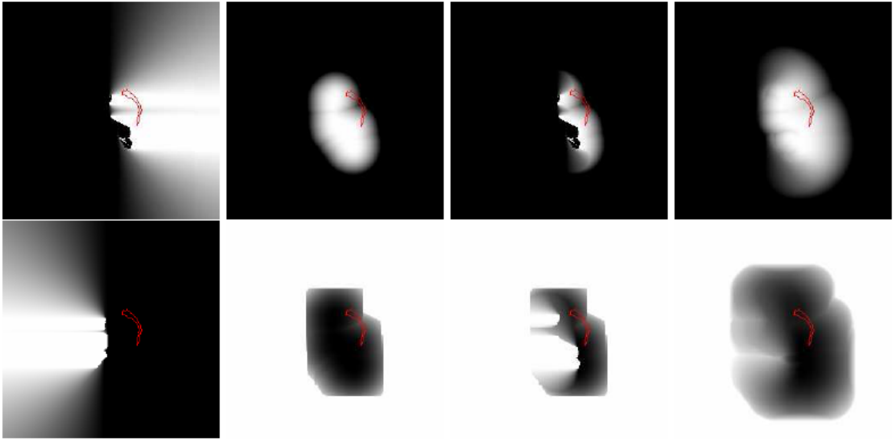
$$(\mu_{Fusion}, \nu_{Fusion}) = (\min(\mu_{dir}, \mu_{dist}), \max(\nu_{dir}, \nu_{dist}))$$



Pathological hemisphere: deformations induced by the tumor.

- Semantics of the induced variability: (μ_{var}, ν_{var})
- Larger region, including the correct region:

$$(\mu'_{dist}, \nu'_{dist}) = \delta_{(\mu_{var}, \nu_{var})}(\mu_{dist}, \nu_{dist})$$



Conclusion

- Algebraic framework of mathematical morphology.
- Strong properties.
- Local and structural knowledge representation and reasoning.
- Applies in different frameworks (logics, fuzzy sets, bipolarity, FCA...).
- Towards spatial reasoning.
- Towards preference modeling and decision making.
- Extension to spatio-temporal reasoning?