Classification of the Perspective-Three-Point Problem, Discriminant Variety and Real Solving Polynomial Systems of Inequalities

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ABSTRACT

Classifying the Perspective-Three-Point problem (abbreviated by P3P in the sequel) consists in determining the number of possible positions of a camera with respect to the apparent position of three points. In the case where the three points form an isosceles triangle, we give a full classification of the P3P. This leads to consider a polynomial system of polynomial equations and inequalities with 4 parameters which is generically zero-dimensional. In the present situation, the parameters represent the apparent position of the three points so that solving the problem means determining all the possible numbers of real solutions with respect to the parameters' values and give a sample point for each of these possible numbers. One way for solving such systems consists first in computing a *discriminant variety*. Then, one has to compute at least one point in each connected component of its real complementary in the parameter's space. The last step consists in specializing the parameters appearing in the initial system by these sample points. Many computational tools may be used for implementing such a general method, starting with the well known Cylindrical Algebraic Decomposition (CAD in short), which provides more information than required. In a first stage, we propose a full algorithm based on the straightforward use of some sophisticated software such as FGb (Gröbner bases computations) RS (real roots of zero-dimensional systems), DV (Discriminant varieties) and RAGlib (Critical point methods for semi-algebraic systems). We then improve the global algorithm by refining the required computable mathematical objects and related algorithms and finally provide the classification. Three full days of computation were necessary to get this classification which is obtained from more than 40000 points in the parameter's space.

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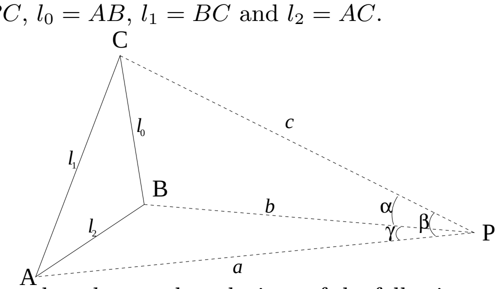
1. INTRODUCTION

The perspective-n-points problem has been widely studied during the past decades [7, 13, 30, 9]. The goal is to determine the position of a camera given the apparent position of n points. This problem has many applications in fields such as computer vision [16] or automated cartography [7] for example. It was summarized in [7] for the general case of n points as follow:
"Given the relative spatial location of n control points, and given the angle to every pair of control points from an additional point called the Center of Perspective (CP), find the lengths of the line segments joining CP to each of the control points."
It was proved in 1984 [8] that for $n \geq 6$, the position of the Center of Perspective is uniquely determined by the angles from CP to the pairs of control points. Different algorithms were designed for the direct computation. Harlick et al. reviewed the major direct solutions before 1991 in [13]. More recently, this direct problem has been revisited in [21], [1], [22] and references therein.
However, in [7] Fischler and Bolles observed that the P3P problem could have from 0 up to 4 solutions, according to the triangle lengths and the angles observed from the perspective point of view. Moreover, in [15] and references therein, Z.Y. Wu and F.C. Hu show that the P5P problem can have 2 solutions, and the P4P problem up to 5 solutions. But for

each number of solutions, the corresponding set of parameters was not given.
In [10], X.-S. Gao and J. Tang have finally proved that for $n \geq 4$, the set of parameters where the PnP problem admits more than one solution has a zero volume.
In [30] and [9] Yang, Gao and al. exhibit respectively a partial and an exhaustive set of polynomial conditions to decide the number of solutions of the P3P problem. These conditions were obtained by combining a triangular decomposition, resultant computations and a careful use of the Descartes' rule of sign and of the Sylvester-Habicht sequences. However, these conditions are rather complex and do not give usable informations on the geometry of these cells. In particular, testing the satisfiability of such conditions is challenging since it is not feasible with current CAD and generic CAD software.
Recently in [31], [29], the authors give some geometrical condition for the P3P problem to have 4 solutions, and provide a guide to arrange control points in real applications.
In this article, we present an efficient and certified method to obtain a more intuitive classification of the parameters of a system with respect to its number of solutions. We then apply this method to the P3P problem in the case where the three points form an isosceles triangle. Our output will be a so-called *open classification* . In particular, *for each possible number of solutions of the perspective three point problem we provide at least one point in each component of the corresponding set of parameters.* Our classification is said open because we ensure that the components we compute have a non-null volume and are thus reachable in practice. In particular we omit components of null volume, since they have no physical interest.
Main result and related work. The complete resolution of the P3P problem is given for an isosceles triangle. It required 3 days of computations. The number of computed points is 60086.
A first key tool which allowed us to solve this problem is the *discriminant variety.* This notion is related to the implicit function theorem and has many variants, as those presented in [19, 11] for example. We use the definition of [17], where a Discriminant Variety of a parametric system with rational or real coefficients is an algebraic variety of the parameter's space such that, among other remarkable properties, the number of real solutions of the initial system is invariant on each connected component of its complementary over the reals. For its computation, we used the algorithm provided in [17] and implemented in the DV package for the so-called *well-behaved systems.*
CAD implementations could not compute at least one point in each connected component of the complementary of the discriminant variety over the reals, due to the prohibitive number of cells (and timing) induced by the doubly exponential behaviour of the algorithms. Thus we developed new algorithms to tackle this problem. These algorithms are based on the critical point method. These methods are known to have a better worst-case complexity than CAD algorithm (see [3, Chapter 13] and references therein). Nevertheless, up to now, they used infinitesimals [2, 12, 14] which makes heavier the arithmetic with which computations are performed and, finally, spoil the practical behavior of this method. The first algorithm we developed, generalizes the strategy developed in [26] and computes *a priori* specialization values for the infinitesimal. This algorithm allowed us to solve the problem but required 3 weeks of computations. Then, we developed a second algorithm whose practical behavior is much better, since this one allows us to solve completely the problem in 3 days.
In the first two parts, we will recall the P3P problem and present our general classification strategy. Then we will detail the two main algorithms of our method. Finally, we will show its application on the P3P problem.

2. DESCRIPTION OF THE PERSPECTIVE THREE POINT PROBLEM

Let A,B and C be the three control points, P be the perspective point and α,β, γ be respectively the three angles $\widehat{BPC}$,$\widehat{APC}$ and $\widehat{APB}$. Furthermore, let $a = PA$, $b = PB$, $c = PC$, $l_0 = AB$, $l_1 = BC$ and $l_2 = AC$.



The lengths a,b,c are the solutions of the following equations given in [7]:

$$\begin{cases} l_0^2 &= a^2 + b^2 - 2ab\cos(\alpha) \\ l_1^2 &= b^2 + c^2 - 2bc\cos(\beta) \\ l_2^2 &= a^2 + c^2 - 2ac\cos(\gamma) \end{cases}$$

We denote by u,v and w the expressions $2\cos(\alpha)$, $2\cos(\beta)$ and $2\cos(\gamma)$. Moreover, as in [9], we express all the lengths of our problem relatively to l_0. Thus we introduce $A = \frac{a}{l_0}$, $B = \frac{b}{l_0}$, $C = \frac{c}{l_0}$. And for the length of the triangle, we use $x = \frac{l_2^2}{l_0^2}$ and $t = \frac{l_1^2}{l_0^2}$. Thus we get the following simplified system:

$$\begin{cases} 1 &= A^2 + B^2 - ABu \\ t &= B^2 + C^2 - BCv \\ x &= A^2 + C^2 - ACw \end{cases}$$

with the following constraints:

$$x > 0, t > 0, -2 < u < 2, -2 < v < 2, -2 < w < 2$$

where:
- A, B, C are the *unknowns*
- x, t, u, v, w are the *parameters*
We will present a general method to classify the parameters of such a system. Given a number k, this method allows us to say if there exists an open set of the parameters where the system admit exactly k solutions.
We will show the application of this method for the classification of the parameters of the P3P problem, in the case where the triangle is isosceles.

3. CLASSIFICATION METHOD - DISCRIMINANT VARIETY

Goal. Let $S_{\mathbf{T}}(\mathbf{X})$ be a parametric system of polynomial equalities and inequalities in $\mathbb{Q}[\mathbf{T}][\mathbf{X}]$, where $\mathbf{T} = T_1, \ldots, T_s$

are the parameters and $\mathbf{X} = X_1, \ldots, X_n$ the unknowns. We want to be able to answer to the following question:
"Given a parametric system $S_{\mathbf{T}}$ and an integer i, does there exist an open set $\mathcal{O}$ in the parameters' space such that for all $p_0 \in \mathcal{O}$, the number of solutions of S_{p_0} is i? If yes, give explicitly a point $a \in \mathcal{O}$".
For this purpose, we present a method to classify the parametric values p_0 of a dense open set of $\mathcal{P}$ according to the number of real solutions of S_{p_0}. In the following, $\mathcal{P}$ will denote the real parameters' space. The method we describe in this article computes exactly an *open classification* of $\mathcal{P}$ with relation to $S_{\mathbf{T}}$ according to the following definition:

Definition 1. (Open classification) *Let $S_{\mathbf{T}}(\mathbf{X})$ be a parametric system. Let $k \in \mathbb{N}$ and $\mathcal{O}_0, \ldots, \mathcal{O}_k$ be open sets (for the euclidean topology) in the parameters' space such that:*

$$\begin{cases} \forall p_0 \in \mathcal{O}_i, S_{p_0} \text{ has } i \text{ real solutions} \\ \bigcup_{i=0}^{k} \mathcal{O}_i \text{ is dense in the parameters' space} \end{cases}$$

We call the family $(\mathcal{O}_i)_{0 \le i \le k}$ an open classification *of $\mathcal{P}$ with relation to $S_{\mathbf{T}}$.*
As announced in the introduction, the proposed methods will be based on the discriminant variety introduced in [17].

Definition 2. *(Discriminant variety) Given a constructible set $\mathcal{C}$, a discriminant variety of $\mathcal{C}$ is an algebraic set in the parameter's space such that a restriction of the trivial projection from $\mathcal{C}$ onto the complementary of the discriminant variety in the parameters' space defines an analytic cover.*

In addition, a discriminant variety is the parameters' space itself if and only if each of the (complex) fibers are infinite.

Definition 3. *(Minimal discriminant variety) The minimal discriminant variety is the intersection of all the discriminant varieties (and is thus a discriminant variety).*

Remark 1. *In particular, the complementary of a discriminant variety defines an open classification of $\mathcal{P}$ with relation to $S_{\mathbf{T}}$.*

Computing an open classification.

Given a parametric system $S_{\mathbf{T}}$, we show that an *open classification* of $S_{\mathbf{T}}$ can be represented by (q, F, ϕ), which are defined as follows:

- q is a polynomial and a discriminant variety of $S_{\mathbf{T}}$;
- F a set of rational points in each connected component of $q \neq 0$;
- ϕ is a table which associates to each point p_0 of F the number of solutions of the 0-dimensional systems S_{p_0}.

In this representation, each $\mathcal{O}_i$ is represented by q and the subset of points $\phi^{-1}(i) \subset F$ such that:

$$\mathcal{O}_i = \{\ x \in \mathcal{P} \mid \text{ there exists } p \in \phi^{-1}(i) \text{ and a continuous path from } p \text{ to } x \text{ included in } q \neq 0\}$$

To compute this representation, our algorithm is naturally decomposed in three steps:
Input: a parametric system $S_{\mathbf{T}}$, the set of parameters $\mathbf{T}$, and the set of unknowns $\mathbf{X}$.
Output: the 3-tuple (q, F, ϕ)
Main algorithm:
Step a: The discriminant variety q. For the first step, we compute q as a polynomial vanishing at the discriminant variety of $S_{\mathbf{T}}$. The full algorithm may be found in [17] and the main ideas of its computation are recalled in the appendix. It is implemented in the maple DVlib package and will be directly available in the next release of Maple [18].
Step b: The sampling points F. The critical point method allows to compute at least one point in each connected component of a semi-algebraic set defined by strict inequalities. An algorithm using these methods is given in [26]. We show in section 4 an improvement of this algorithm. In this step, F is a finite set of point in each connected component of the semi-algebraic set defined by $q \neq 0$. This function is implemented in the maple RAGlib package.
Step c: The table ϕ Finally, we compute a table where each point p_0 of F is associated to the number of real solutions of the system S_{p_0}. For this step, we use the Rational Univariate Representation presented in [23] and implemented in the RS software which gives a list of non overlapping boxes with rational bounds, containing the real solutions of a zero-dimensional system. □
Theoretically, the first step has the largest complexity upper bound. However, in practice the behavior of the three steps does not follow the same scheme. In particular, the first step is not often slower than the other steps.

4. SOLVING SYSTEMS OF POLYNOMIAL INEQUALITIES

As described above, once a discriminant variety V is computed, one has to compute at least one point in each connected component of $\mathbb{R}^n \setminus V$. RAGlib provides routines allowing us to tackle this computation. This section is devoted to present the algorithms we implemented in RAGlib and give some sketch of proofs. The techniques we use are based on computations of critical points or critical values of polynomial mappings. In the whole section, we consider a polynomial family $(f_1, \ldots, f_s)$ in $\mathbb{Q}[X_1, \ldots, X_n]$ of degree bounded by D. We denote by $\mathcal{S}$ the semi-algebraic set defined by $f_1 > 0, \ldots, f_s > 0$ *which is supposed to be bounded*. Denote by $\mathbb{Q}_+$ the set of positive rationals. Given $\mathbf{a} = (a_1, \ldots, a_s) \in \mathbb{Q}_+^s$, $\mathcal{I} = \{i_1, \ldots, i_\ell\} \subset \{1, \ldots, s\}$, and $e \in \mathbb{R}$ (resp. E a new variable) we denote by $S_{\mathcal{I},e}^{(\mathbf{a})}$ (resp. $S_{\mathcal{I},E}^{(\mathbf{a})}$) the polynomial system $f_{i_1} - a_{i-1}e = \cdots = f_{i_\ell} - a_{i_\ell}e = 0$ (resp. $f_{i_1} - a_{i_1}E = \cdots = f_{i_\ell} - a_{i_\ell}E = 0$) and by $V_{\mathcal{I},e}^{(\mathbf{a})} \subset \mathbb{C}^n$ (resp. $V_{\mathcal{I},E}^{(\mathbf{a})} \subset \mathbb{C}^{n+1}$) the algebraic variety it defines. In the sequel, we use the following notations:

– Π denotes the canonical projection $(x_1, \ldots, x_n, e) \in \mathbb{C}^{n+1} \to (x_1, \ldots, x_n) \in \mathbb{C}^n$;
– The canonical projection $(x_1, \ldots, x_n) \in \mathbb{C}^n \to x_i \in \mathbb{C}$ (resp. $(x_1, \ldots, x_n, e) \in \mathbb{C}^{n+1} \to e \in \mathbb{C}$) is denoted by π_i (resp. π_E).
– Given a polynomial mapping $\varphi : \mathbb{C}^n \to \mathbb{C}^k$ and an algebraic variety $V \subset \mathbb{C}^n$, $\mathfrak{C}(\varphi, V)$ denotes the critical locus of the restriction of φ to V;
– Given a polynomial mapping $\varphi : \mathbb{C}^n \to \mathbb{C}^k$ and an algebraic variety $V \subset \mathbb{C}^n$, $\mathcal{K}(\varphi, V)$ denotes the critical values of the restriction of φ to V;
– Given $\mathcal{I}$ and $\mathbf{a}$, $\mathfrak{C}_{\mathcal{I}}^{\mathbf{a}}$ denotes the Zariski-closure of

$$\cup_{e \in \mathbb{C} \setminus \mathcal{K}(\pi_1, V_{\mathcal{I},E}^{(\mathbf{a})})} \{(x, e) \in \mathbb{C}^{n+1} \mid x \in \mathfrak{C}(\pi_1, V_{\mathcal{I},e}^{(\mathbf{a})})\}$$

– Given $\mathcal{I}$ and $\mathbf{a}$, $e^{\mathbf{a}}_{\mathcal{I}}$ denotes the smallest positive value of $\mathcal{K}(\pi_E, V^{(\mathbf{a})}_{\mathcal{I},E}) \bigcup_{j\in\{1,\dots,s\}\setminus\mathcal{I}} \pi_E(\mathfrak{C}^{\mathbf{a}}_{\mathcal{I}} \cap \{x\in\mathbb{R}^n \mid f_j(x) = 0\})$.

The following result shows that under some assumptions of genericity on $\mathbf{a}$ and e, $V^{(\mathbf{a})}_{\mathcal{I},e}$ is smooth and equi-dimensional, and the ideal generated by $S^{(\mathbf{a})}_{\mathcal{I},e}$ is radical. These properties are useful to compute critical points of polynomial mappings restricted to $V^{(\mathbf{a})}_{\mathcal{I},e}$ by intersecting $V^{(\mathbf{a})}_{\mathcal{I},e}$ with the vanishing set of some minors of the Jacobian matrix associated to $S^{(\mathbf{a})}_{\mathcal{I},e}$.

LEMMA 1. *Given $\mathcal{I} \subset \{1,\dots,s\}$, there exists a proper Zariski-closed subset $\mathcal{A}\times\mathcal{E} \subsetneq \mathbb{C}^s\times\mathbb{C}$ such that for all $\mathbf{a}\in\mathbb{Q}^s\setminus\mathcal{A}$ and $e\in\mathbb{Q}\setminus\mathcal{E}$, $S^{(\mathbf{a})}_{\mathcal{I},e}$ generates a radical and equidimensional ideal and $V^{(\mathbf{a})}_{\mathcal{I},e}$ is smooth of dimension $n-\sharp\mathcal{I}$ or empty.*

PROOF. Consider the polynomial mapping $\varphi : x\in\mathbb{C}^n \to (f_{i_1}(x),\dots,f_{i_\ell}(x)) \in \mathbb{C}^\ell$. From the algebraic Sard's theorem [3], the set of critical values of φ is contained in a proper Zariski-closed subset $\mathcal{A}\subsetneq\mathbb{C}^\ell$. This implies that for all $(a_{i_1},\dots,a_{i_\ell})\in\mathbb{Q}^\ell\setminus\mathcal{A}$, the system $f_{i_1}-a_{i_1}=\cdots=f_{i_\ell}-a_{i_\ell}=0$ generates a radical and equidimensional ideal and defines a smooth variety of dimension $n-\ell$. Considering now the restriction of the mapping $(x,e)\in\mathbb{C}^n\times\mathbb{C}\to e$ to the variety defined by $f_{i_1}-a_{i_1}e=\cdots=f_{i_\ell}-a_{i_\ell}e=0$ and using as previously, Sard's theorem ends the proof. □

First algorithm. This paragraph contains a description of a first algorithm which generalizes to the case of polynomial systems of inequalities the strategy developed in [26]. From the proposition below, one can reduce the computation of at least one point in each connected component of $\mathcal{S}$ to the computation of at least one point in each connected component of real algebraic sets $V^{(\mathbf{a})}_{\mathcal{I},e}\cap\mathbb{R}^n$ if e is small enough.

PROPOSITION 1. *Let C be a connected component of $\mathcal{S}$. There exists $\mathcal{I}\subset\{1,\dots,s\}$ and $e_0>0$ such that for all $e\in]0,e_0[$, there exists a connected component of $V^{(\mathbf{a})}_{\mathcal{I},e}\cap\mathbb{R}^n$ contained in C.*

PROOF. The result is an immediate application of the transfer principle and [3, Chapter 13, Proposition 13.2], remarking that $\mathcal{S}$ is also defined by $\frac{f_1}{a_1}>0,\dots,\frac{f_s}{a_s}>0$. □

Given $\mathbf{a}\in\mathbb{Q}^s$, a connected component C of $\mathcal{S}$, and $\mathcal{I}$ such that the first item of Proposition 1 above applies, we show below how to compute e_0.

THEOREM 1. *Let C be a connected component of $\mathcal{S}$. There exists $\mathcal{I}\subset\{1,\dots,s\}$ such that for all $e\in]0,e^{\mathbf{a}}_{\mathcal{I}}[$ there exists a point of $\mathfrak{C}(\pi_1, V^{(\mathbf{a})}_{\mathcal{I},e})$ lying in C;*

PROOF. Let $\mathbf{a}=(a_1,\dots,a_s)\in\mathbb{Q}^s$. Consider a maximal (for the order inclusion) subset $\mathcal{I}=\{i_1,\dots,i_\ell\}\subset\{1,\dots,s\}$ such that Proposition 1 applies. Then, there exists $e_0>0$ such that for $e\in]0,e_0[$ there exists a compact connected component C_e of $V^{(\mathbf{a})}_{\mathcal{I},e}\cap\mathbb{R}^n$ included in C. Since C_e is compact, it contains a point x_e of $\mathfrak{C}(\pi_1,V^{(\mathbf{a})}_{\mathcal{I},e})\subset\mathfrak{C}^{\mathbf{a}}_{\mathcal{I}}$. Denote by B a ball containing C. Since $e^{\mathbf{a}}_{\mathcal{I}}$ is less than or inferior to the minimum of the positive real numbers of $\mathcal{K}(\pi_E,V^{(\mathbf{a})}_{\mathcal{I},E})$, the Ehresmann's fibration theorem implies that Π_E realizes a locally trivial fibration over $\{(x,e)\in B\times\mathbb{R}\}\cap V^{(\mathbf{a})}_{\mathcal{I},E}\cap\pi_E^{-1}(]0,e^{\mathbf{a}}_{\mathcal{I}}[)$. This implies that C_e varies continuously and remains non-empty and compact when e varies in $]0,e^{\mathbf{a}}_{\mathcal{I}}[$ and then has a non-empty intersection with $\mathfrak{C}(\pi_1,V^{(\mathbf{a})}_{\mathcal{I},e})$. Note at last that by definition of $e^{\mathbf{a}}_{\mathcal{I}}$, for all $j\in\{1,\dots,s\}\setminus\mathcal{I}$, f_j is positive at each point of this intersection. □

Given $\mathbf{A}\in GL_n(\mathbb{Q})$ and $f\in\mathbb{Q}[X_1,\dots,X_n]$, we denote by $f^{\mathbf{A}}$ the polynomial $f(\mathbf{AX})$ where $\mathbf{X}=[X_1,\dots,X_n]$. If $V\subset\mathbb{C}^n$ is an algebraic variety defined by $f_1=\cdots=f_s=0$ (with $\{f_1,\dots,f_s\}\subset\mathbb{Q}[X_1,\dots,X_n]$, $V^{\mathbf{A}}$ denotes the variety defined by $f_1^{\mathbf{A}}=\cdots=f_s^{\mathbf{A}}=0$.

PROPOSITION 2. *Let $e\in\mathbb{C}\setminus\mathcal{K}(\pi_E,V^{(\mathbf{a})}_{\mathcal{I},E})$. There exists a Zariski-closed subset $\mathcal{A}\subsetneq GL_n(\mathbb{C})$ such that for all $\mathbf{A}\in GL_n(\mathbb{Q})\setminus\mathcal{A}$, $\mathfrak{C}(\pi_1,{V^{(\mathbf{a})}_{\mathcal{I},e}}^{\mathbf{A}})$ is either empty or zero-dimensional.*

PROOF. Since $e\notin\mathcal{K}(\pi_E,V^{(\mathbf{a})}_{\mathcal{I},E})$, $V^{(\mathbf{a})}_{\mathcal{I},e}$ is smooth. Now, the result is an immediate consequence of [27, Theorem 2]. □

Finally, the algorithm consists in considering all the polynomial systems $S^{(\mathbf{a})}_{\mathcal{I},E}$, compute a univariate polynomial whose set of roots is $\mathcal{K}(\pi_E,V^{(\mathbf{a})}_{\mathcal{I},E})$, find a rational number $e>0$ less than the smallest positive root of this polynomial and compute a rational parameterization of $\mathcal{K}(\pi_1,V^{(\mathbf{a})}_{\mathcal{I},e})$ up to a generic enough linear change of variables. All these computations are done using algebraic elimination routines. In practice RAGLIB uses Gröbner bases FGB engine [5, 6].

REMARK 2. *In order to generalize this strategy to the general case of non-bounded semi-algebraic sets, it is sufficient to compute the set of* generalized critical values *of π_E restricted to $V^{(\mathbf{a})}_{\mathcal{I},E}$ (see [26, 25] for algorithms performing such computations), find a rational number $e>0$ less than the smallest generalized critical value, and to* compute at least one point in each connected component *of $V^{(\mathbf{a})}_{\mathcal{I},e}$ (see [27] for an efficient algorithm performing such a task).*

Second algorithm. We will see in the next section that the results (in terms of computation timings) obtained by the above algorithm are not satisfactory even if it has allowed us to solve completely the classification problem we consider here. This paragraph is devoted to design an other algorithm, based on similar techniques than those developed above, which is more efficient. It avoids the computations of $\mathcal{K}(\pi_E,V^{(\mathbf{a})}_{\mathcal{I},E})$ which are expensive, in particular when the critical locus $\mathfrak{C}(\pi_E,V^{(\mathbf{a})}_{\mathcal{I},E})$ has not dimension 0. This is the case in the application we consider. Given a connected component C of $\mathcal{S}$, there exists $\mathcal{I}\subset\{1,\dots,s\}$ from Proposition 1 such that the real algebraic set $V^{(\mathbf{a})}_{\mathcal{I},e}$ has a connected component contained in C *for e small enough.* The idea we develop here is the following. Instead of computing a specialization value for e (see Theorem 1 above), we focus on the informations one can get when e tends to 0. From Proposition 2, there exists a Zariski-closed subset $\mathcal{E}\subsetneq\mathbb{C}$ such that for all $e\in\mathbb{R}\setminus\mathcal{E}$, up to a generic linear change of coordinates, $\mathfrak{C}(\pi_1,V^{(\mathbf{a})}_{\mathcal{I},e})$ is either empty or zero-dimensional. In the sequel, we denote by $\mathfrak{C}_1^{\mathcal{I}}$ the Zariski-closure of $\cup_{e\in\mathbb{C}\setminus\mathcal{E}}\mathfrak{C}(\pi_1,V^{(\mathbf{a})}_{\mathcal{I},e})$. From Hardt's semi-algebraic triviality theorem, there exists $e_0\in\mathbb{R}$ such that $\mathfrak{C}(\pi_1,V^{(\mathbf{a})}_{\mathcal{I},e})\times]0,e_0[$ is homeomorphic $\cup_{e\in]0,e_0[}\mathfrak{C}(\pi_1,V^{(\mathbf{a})}_{\mathcal{I},e})$. Thus, one can define by continuity the set $\lim_0\mathfrak{C}(\pi_1,V^{(\mathbf{a})}_{\mathcal{I},e})$ of finite limits of $\mathfrak{C}(\pi_1,V^{(\mathbf{a})}_{\mathcal{I},e})$ when e tends to 0. In particular, we look at $\lim_0\mathfrak{C}(\pi_1,V^{(\mathbf{a})}_{\mathcal{I},e})$ and $\lim_0\mathcal{K}(\pi_1,V^{(\mathbf{a})}_{\mathcal{I},e})$.

THEOREM 2. *Consider C be a connected component of the bounded semi-algebraic set $\mathcal{S}$. There exists $\mathcal{I} \subset \{1,\dots,s\}$ such that*
a) given $\{\xi_1,\dots,\xi_k\} = \lim_0(\mathcal{K}(\pi_1, V^{(\mathbf{a})}_{\mathcal{I},e}))$ (with $\xi_1 < \dots < \xi_k$),
b) and $r_1,\dots,r_{k-1}$ be rationals such that $\xi_1 < r_1 < \dots < r_{k-1} < \xi_k$,
there exists $i \in \{1,\dots,k-1\}$ for which $C \cap \pi_1^{-1}(r_i) \neq \emptyset$.

PROOF. From Proposition 1, there exists $\mathcal{I} \subset \{1,\dots,s\}$ a maximal set (for the order inclusion) and $e_0 > 0$ such that for $e \in]0, e_0[$, $V^{(\mathbf{a})}_{\mathcal{I},e} \cap \mathbb{R}^n$ has a connected component C_e included in C.
Since C is bounded, C_e is compact. Thus, its image by the projection π_1 is a closed interval $[a_e, b_e]$. We prove now that when e tends to 0, a_e (resp. b_e) has a finite limit in $\mathbb{R}$ denoted by a_0 (resp. b_0). Since $\mathcal{S}$ is supposed to be bounded, there exists a ball $B \subset \mathbb{R}^n$ such that $C \subset B$. Suppose now that a_e has not a finite limit in $\mathbb{R}$ when e tends to 0. This implies that there exists e small enough such that C_e is not contained in B while it is still contained in C. This is a contradiction. Consider now $r \in]a_0, b_0[$. There obviously exists $e > 0$ such that $a_e < r < b_e$. Thus $C \cap \pi_1^{-1}(r) \neq \emptyset$. □
Denote by $\mathrm{Jac}(f_{i_1},\dots,f_{i_\ell})$ the Jacobian matrix associated to $f_{i_1},\dots,f_{i_\ell}$.

$$\begin{bmatrix} \frac{\partial f_{i_1}}{\partial X_1} & \dots & \frac{\partial f_{i_1}}{\partial X_n} \\ \vdots & \vdots & \vdots \\ \frac{\partial f_{i_\ell}}{\partial X_1} & \dots & \frac{\partial f_{i_\ell}}{\partial X_n} \end{bmatrix}$$

Denote by $\Sigma_{\mathcal{I}}$ the set of all $(n-\ell, n-\ell)$ minors of $\mathrm{Jac}(f_{i_1},\dots,f_{i_\ell})$. We consider in the sequel $\Delta_{\mathcal{I}}$ the set of $(n-\ell, n-\ell)$ minors of the matrix obtained after removing the first column of $\mathrm{Jac}(f_{i_1},\dots,f_{i_\ell})$ and $\delta_{\mathcal{I}} = \Sigma_{\mathcal{I}} \setminus \Delta_{\mathcal{I}}$. In the sequel, $\delta_{\mathcal{I}}[i]$ denotes the i-th element of $\delta_{\mathcal{I}}$ and $\delta^{(i)}_{\mathcal{I}}$ denotes the first i elements of $\delta_{\mathcal{I}}$. Given $\mathcal{I} = \{i_1,\dots,i_\ell\} \subset \{1,\dots,s\}$ and $\mathbf{a} = (a_1,\dots,a_s) \in \mathbb{Q}^s$, we consider polynomial systems $\mathcal{S}^{\mathbf{a}}_{\mathcal{I}}$ $a_{i_1} f_{i_2} - a_{i_2} f_{i_1} = \dots = a_{i_1} f_{i_\ell} - a_{i_\ell} f_{i_1} = 0$ and the ideals defined by:

$$I_i = \langle \mathcal{S}^{\mathbf{a}}_{\mathcal{I}}, L\delta_{\mathcal{I}}[i+1] - 1, \delta^{(i)}_{\mathcal{I}}, \Delta_{\mathcal{I}} \rangle \cap \mathbb{Q}[X_1,\dots,X_n]$$

The following result shows how to compute $\lim_0(\mathcal{K}(\pi_1, V^{(\mathbf{a})}_{\mathcal{I},e}))$ under assumptions of genericity on $\mathbf{a}$. Its proof is based on similar techniques than the ones used in [24].

THEOREM 3. *There exists a Zariski-closed subset $\mathcal{A} \times \mathcal{E} \subsetneq \mathbb{C}^n \times \mathbb{C}$ such that for all $\mathbf{a} \in \mathbb{Q}^s \setminus \mathcal{A}$, $e \in \mathbb{R} \setminus \mathcal{E}$ and for all $\mathcal{I} \subset \{1,\dots,s\}$, $S^{(\mathbf{a})}_{\mathcal{I},\varepsilon}$ generates a radical and equidimensional ideal and $V^{(\mathbf{a})}_{\mathcal{I},e}$ is smooth of dimension $n - \sharp\mathcal{I}$ if it is not empty. Then, $\lim_0(\mathcal{K}(\pi_1, V^{(\mathbf{a})}_{\mathcal{I},e}))$ belongs to the union of the sets of solutions of $(I_i + \langle f_{i_1} \rangle) \cap \mathbb{Q}[X_1]$ for $i = 0$ to $\sharp\delta_{\mathcal{I}} - 1$.*

Thus, the algorithm consists, for all $\mathcal{I} \subset \{1,\dots,s\}$, in computing $\lim_0(\mathcal{K}(\pi_1, V^{(\mathbf{a})}_{\mathcal{I},e}))$ which is represented by a univariate polynomial. Then, it isolates the real roots of these polynomials and find one rational number r between each successive isolated real root. Then, it performs recursively, substituting X_1 by r in the input polynomials.
Note the difference with CAD algorithm: here we obtain boundary points of the projections of the connected components of the studied semi-algebraic set by computing directly limits of critical points. This allows to avoid the growth of degree and the appearance of superfluous values induced by the recursive projection step of CAD. Note also that we never compute the critical loci $\mathcal{C}(\pi_E, V^{(\mathbf{a})}_{\mathcal{I},E})$ considered in the algorithm described in the previous paragraph.

REMARK 3. *If $V^{\mathbf{a}}_{\mathcal{I},0}$ is smooth and $S^{\mathbf{a}}_{\mathcal{I},0}$ generates a radical equidimensional ideal, $\lim_0(\mathcal{K}(\pi_1, V^{(\mathbf{a})}_{\mathcal{I},e}))$ is contained in $\mathcal{K}(\pi_1, V^{\mathbf{a}}_{\mathcal{I},0})$.*

Given $\mathcal{I} \subset \{1,\dots,s\}$, note that the computations performed by the algorithm we present in this paragraph can be seen as computing $\mathfrak{C}^{\mathbf{a}}_{\mathcal{I}} \cap \{(x,0) \in \mathbb{C}^{n+1}\}$. The following result shows that the degree of the curve $\mathfrak{C}^{\mathbf{a}}_{\mathcal{I}}$ is well-controlled and is related to the positive dimensional components of $\mathfrak{C}(\pi_E, V^{(\mathbf{a})}_{\mathcal{I},E})$.

PROPOSITION 3. *The degree of $\mathfrak{C}^{\mathbf{a}}_{\mathcal{I}}$ is bounded by $D^\ell (D-1)^{n-\ell}\binom{n-1}{\sharp\mathcal{I}} - \delta_{\mathrm{sing}}$ where δ_{sing} is the sum of the degrees of the positive irreducible components of $\mathfrak{C}(\pi_E, V^{(\mathbf{a})}_{\mathcal{I},E})$.*

Its proof uses a similar reasoning than those in [24] and bi-homogeneous bounds computed from a Lagrangian formulation of critical points (see [28]).

REMARK 4. *When $\mathcal{S}$ is not supposed to be bounded, the above algorithm has to be modified to compute sampling points of $\mathcal{S}$ by substituting the computation of $\lim_0(\mathcal{K}(\pi_1, V^{(\mathbf{a})}_{\mathcal{I},e}))$ by the computation of limits of* generalized critical values *of π_1 restricted to $V^{(\mathbf{a})}_{\mathcal{I},e}$ when e tends to 0.*

5. COMPUTATIONS AND RESULTS

We show here the results of the computations we obtained solving the P3P problem. We do the computation by restriction to the case where the triangle we observe is isosceles, that is: $l_0 = l_1$. The system we consider is:

$$\begin{cases} 1 &= A^2 + B^2 - ABu \\ 1 &= B^2 + C^2 - BCv \\ x &= A^2 + C^2 - ACw \end{cases}$$

It has 4 parameters u,v,w,x and 3 unknowns A,B,C.
All the computations have been performed on a PC Intel(R) Xeon(TM) CPU 3.20GHz with 6Gb of RAM.

The minimal discriminant variety. We first compute the minimal discriminant variety with the *DV* software in about 1 minute. The result is the polynomial D given in appendix. It is the minimal discriminant variety of the P3P parametric system when the triangle is isosceles. We can notice that D has 7 factors of respective degrees $1, 1, 1, 2, 2, 3, 13$, and whose number of terms is at most 153. Along with the constraints on the parameters, the discriminant variety allows us to define the following semi-algebraic set:

$$D \neq 0, x > 0 - 2 < u < 2, -2 < v < 2, -2 < w < 2$$

The parametric system has a constant number of solutions on each connected component of this semi-algebraic set.

REMARK 5. *The above semi-algebraic set is not bounded in the variable x, which is needed to apply the methods presented in section 4.*

Thus we split this set into $x < 1$ and $x > 1$. Using the variable $y = \frac{1}{x}$, this leads to the study of two bounded semi-algebraic set:

$$\mathcal{H}_x \begin{cases} D \neq 0 \\ 0 < x < 1 \\ -2 < u < 2 \\ -2 < v < 2 \\ -2 < w < 2 \end{cases} \quad \text{and} \quad \mathcal{H}_y \begin{cases} D_y \neq 0 \\ 0 < y < 1 \\ -2 < u < 2 \\ -2 < v < 2 \\ -2 < w < 2 \end{cases}$$

where D_y denotes the polynomial obtained by the substitution of x by $\frac{1}{y}$ in $y^5 D$.

Solving polynomial systems of inequalities. We consider now the two semi-algebraic sets $\mathcal{H}_x$ and $\mathcal{H}_y$. Thanks to the property of the discriminant variety D, we know that on each connected component of these semi-algebraic sets, the parametric system has a constant number of solutions.

To get a *open classification* we first tried to compute a Cylindrical Algebraic Decomposition. However, after one month of computation, we could only complete the projection phase, but not the lifting phase neither with *Maple* nor with *Magma* software. Finally, we implemented the algorithms computing sampling points described in Section 4 in semi-algebraic sets defined by the systems $\mathcal{H}_x$ and $\mathcal{H}_y$. The first algorithm returned a result after 3 weeks of computations, and the second after 3 days. As explained above, this is mainly due to the fact that the discriminant variety contains singularities of high dimension. More generally, we observed that the computation of critical values of the projection π_E considered in Section 5 were particularly difficult. The critical loci of this projection restricted to the varieties considered in Subsection 5.1 have a big dimension. Most of the time spent by the first algorithm described in Subsection 5.1 is spent in these computations. The second algorithm described in Subsection 5.2 avoids the computations of the singularities which appear during the running of the first algorithm. Moreover, as explained in Subsection 5.2, its complexity depends on the *real* geometry of the considered semi-algebraic set. This probably explains why it is so efficient in our case.

These implementations will be soon available in the next release of the RAGLIB Maple package. We successfully got one point in each connected components of $\mathcal{H}_x$ and $\mathcal{H}_y$. As a result we get 13612 points distributed in every connected cell of $\mathcal{H}_x$ and 46474 points in $\mathcal{H}_y$. These points can be downloaded at

`http://www-spiral.lip6.fr/~moroz/P3P.html`

Note that contrarily to polynomials generated randomly, the minimal discriminant variety contains singularities of high dimensions which makes them more difficult to study. Moreover, since D is a minimal discriminant variety , this also ensures us that all conditions on the parameters discriminating the parameters' space according to the number of solutions of the system would contain such singularities.

As we can see on the figures 1 and 2, some connected cells seem very small and almost intractable with random approximations. The drawings show the graph of $\mathcal{H}_x$ around the point p_0, one of the points returned by our computation:

$$\begin{aligned} p_0 &:= (x_0, u_0, v_0, w_0) \\ &= \left(\tfrac{452735729}{9148876946}, \tfrac{3371082457}{1706654848}, \tfrac{2763844376}{1399264123}, \tfrac{26504177576}{13260182015}\right) \\ &\simeq (0.0494853, 1.97525, 1.97521, 1.99877) \end{aligned}$$

On each figure, we present 2 slices centered on p_0. The first figure shows a global view of $\mathcal{H}_x$ and p_0, while the second figure shows a much closer neighborhood of p_0. According

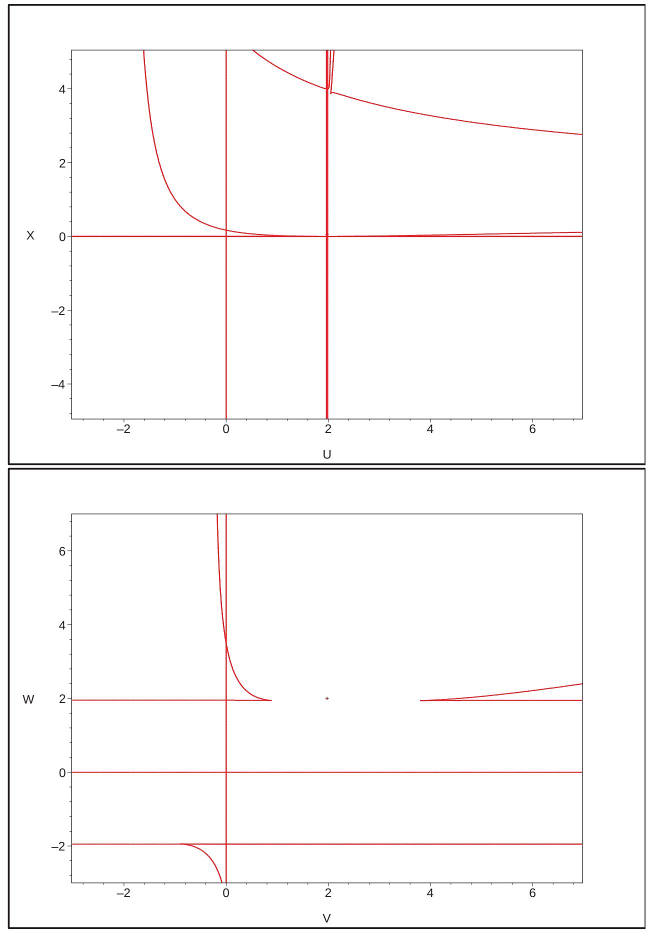


Figure 1: Two slices of $\mathcal{H}$ partly specialized in p_0: - in the first, the variables v and w are specialized - in the second, the variables u and x are specialized

to the slices, we can see that we have detected here a very small connected cell of $\mathcal{H}_x$.

More generally, the set of points we computed intersects each connected component of $\mathcal{H}_x$ and $\mathcal{H}_y$, and we now need to compute the number of solutions of the parametric system specialized in each point to achieve our classification.

Zero-dimensional system solving. In this step, we compute the number of real solutions satisfying the constraint of the problem for 60086 parameters' values. The mean time to solve each corresponding 0-dimensional system is about 0.05 second.

Finally, we can recover the fact that the parametric system of our problem may have exactly $0, 1, 2, 3$ or 4 solutions satisfying the inequalities' constraints. We present in table 1 a sample point in the parameters' space where the system has i solutions for i from 0 to 4.

Moreover, even if we do not have a complete CAD of the discriminant variety, we can have a geometric view of each connected cell of the parameters' space associated to a given number of solutions by drawing the neighborhood of each computed point. As we saw in the previous section, this allowed us for example to exhibit a very small cell, and to compute the number of distinct solutions of the system restricted to this cell, which is exactly 4.

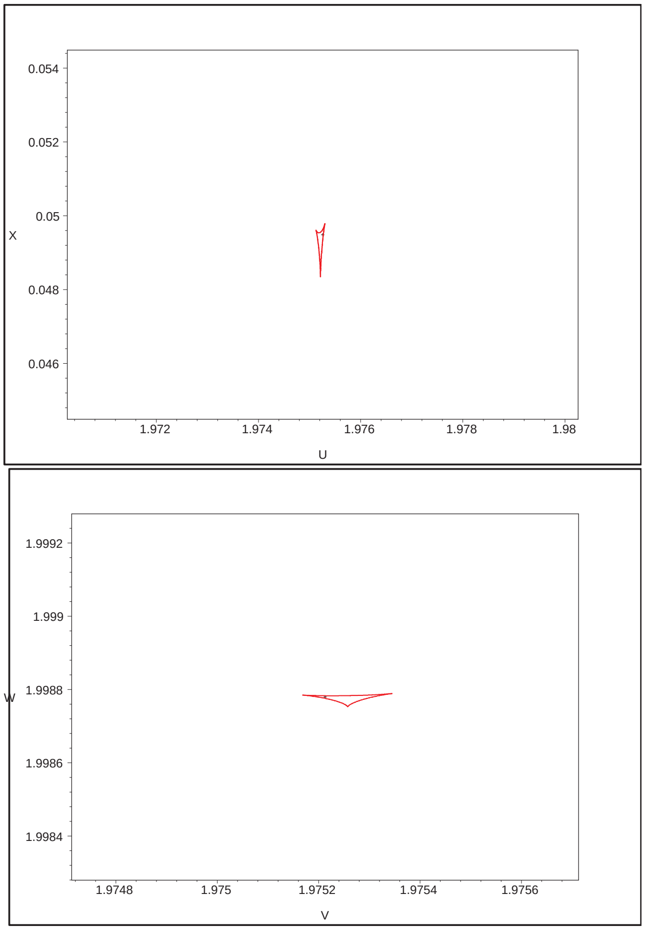


Figure 2: Two closer slices of $\mathcal{H}$ partly specialized in p_0

6. CONCLUSION

We give in this article a full algorithm to compute an intuitive classification of the parameters of a parametric system given by polynomial equations and inequations.
Our method is validated by the description of the parameters' space of the P3P problem when the three control points form an isosceles triangle. We also think that optimizations our implementations and algorithms are possible to make the study of a more general classification.

APPENDIX

Elements of discriminant variety theory

The discriminant variety is presented in [17]. We recall here how to compute it for a *well-behaved* parametric system.

DEFINITION 4. *A parametric system $S_{\mathbf{T}}$ is said well-behaved if and only if:*
i) The number of equations equals the number of unknowns
ii) For all p_0 outside a Zariski closed set, S_{p_0} is radical and zero-dimensional.

The P3P problem and most of the problems coming from applications are modeled by well-behaved systems.
Given a *well-behaved* parametric system $S_{\mathbf{T}}$, let g denote the product of the polynomial inequations of $S_{\mathbf{T}}$. and π the projection map from the solutions of $S_{\mathbf{T}}$ to the parameters' space. If $\mathcal{F}$ is a subset of the parameters' space, then $S_{\mathcal{F}}$

Number of solutions	x	u	v	w
0	$\frac{452735729}{9148876946}$	$-\frac{1087810617}{4897634788}$	$-\frac{2322378129}{10447926511}$	$\frac{4610994663}{2334015862}$
1	$\frac{452735729}{9148876946}$	$-\frac{1087810617}{4897634788}$	$-\frac{2322378129}{10447926511}$	$-\frac{10016606887}{5135366188}$
2	$\frac{452735729}{9148876946}$	$-\frac{1087810617}{4897634788}$	$\frac{2322378129}{10447926511}$	$\frac{10016606887}{5135366188}$
3	$\frac{452735729}{9148876946}$	$-\frac{1087810617}{4897634788}$	$\frac{1270625905}{5709068079}$	$\frac{2776826855}{1423637843}$
4	$\frac{1415953531}{12404789665}$	$\frac{4824522087}{13860411335}$	$\frac{2413516911}{4607583958}$	$\frac{11184766673}{5921669493}$

Table 1: Sample parametric points corresponding to a wanted number of solutions

denotes the restriction of the parametric system $S_{\mathbf{T}}$ to $\mathcal{F}$. The discriminant variety can be decomposed in four algebraic components:
i) V_{ineq} is the projection of the zeros of the polynomial equations and g
ii) V_{sing} is the Zariski closure of the projection of the singular locus of π
iii) V_c is the closure of the critical values of π
iv) V_∞ is the set of parameters' values p_0 such that for all neighborhood $\mathcal{U}_0$ of p_0, the real solutions of $S_{\mathcal{U}_0}$ are not bounded.
The components V_{ineq}, V_{sing}, V_c may be computed by saturation and elimination of variables, which may be handled with Gröbner bases computations (see [4] for example). The component V_∞ may be obtained by extracting some coefficients of a gröbner basis with relation to a block ordering satisfying $\mathbf{X} >> \mathbf{T}$. More details on these computation may be found in [17]. Beside, complexity results of this method are given in [20].

Discriminant variety for the isosceles P3P problem

$$
\begin{aligned}
&D := x(-x+2+w)(x-2+w)\\
&(-x+u^2)(-x+v^2)(-uvw+w^2-4+v^2+u^2)\\
&(-2x^2u^3v^5w^3-72xuv^5w-8u^3v^3w^3-96x^4u^3vw+\\
&6x^2uv^5w^3+4x^3u^4v^4w^2-8x^2u^3v^3w^3+1248xu^2v^2-\\
&24x^3u^4v^2-4x^3u^6w^2-4x^3v^6w^2-24x^3u^2v^4-\\
&96xu^2v^4-128x^5v^2-18x^3u^2v^4w^2-384xu^3vw-\\
&18x^3u^4v^2w^2-12uv^5w^3-96xu^4v^2+24x^3v^4w^2-\\
&240xv^4+576xu^4+x^5u^4v^4-768x^2u^2+\\
&64x^5u^2v^2+576x^2v^4-768x^2v^2+64x^4u^4-\\
&416x^3v^4+64x^3v^6-96x^4uv^3w+256x^4uvw+\\
&48x^2u^3vw^3+8x^2uv^5w+12x^2u^6v^2+168xv^4w^2-\\
&2x^2u^2v^6w^2+12x^2v^6w^2+12x^2u^2v^6-40x^2u^4v^4+\\
&168xu^4w^2+12x^2u^6w^2-8xu^4v^4+16x^5v^4+\\
&xu^6v^2w^2-768x^3uvw+32x^4u^3v^3w-4xu^6v^2-\\
&12xu^6w^2-12xv^6w^2+96xu^3vw^3+16x^2u^4v^4w^2+\\
&8x^4u^5vw+48x^2uv^3w^3+96x^3u^3vw-2x^2u^6v^2w^2+\\
&96x^3uv^3w+60xu^4v^2w^2+96xuv^3w^3+60xu^2v^4w^2+\\
&6xu^2v^2w^4+8x^4uv^5w-336xu^2v^2w^2-384xuv^3w-\\
&2xu^4vw^2-4xu^2v^6-1152x^2u^2v^2-27xu^4w^4-\\
&16x^4u^6-96x^2v^6+64x^3u^6+64xu^6+\\
&64xv^6-240xu^4-128x^5u^2-1024x^4+\\
&1024x^3+768x^2uvw+xu^2v^6w^2+8xu^5v^3w+\\
&6xu^4v^2w^4+48u^5vw+8x^3u^5vw+16x^5u^4-\\
&27xv^4w^4+24x^3u^4w^2+4x^4u^6v+192x^2v^2w^2-\\
&128x^4u^2v^2+4v^6w^2-4x^4u^4v^4+64x^4v^4+\\
&6xu^2v^4w^4+8xu^3v^5w-2xu^5v^3w^3+192x^2u^3vw+\\
&6xu^5vw^3+96u^3v^3w+192x^2uvw+32x^3u^3v^3w-\\
&2xu^3v^5w^3+4u^6w^2+256x^3u^2v^2+96x^3v^2w^2+\\
&48uv^5w-36u^4v^2w^2+256x^4u^2-8x^5u^4v^2-\\
&416x^3u^4+256x^4v^2+384x^3v^2-96x^2u^6-\\
&72xu^5vw-76xu^3v^3w^3+12u^4v^2w^4+4x^4u^2v^6-\\
&12x^3u^6v^2+96x^3u^2w^2-256x^3w^2-8x^5u^2v^4-\\
&16v^6-16x^4v^6+384x^3u^2-16u^6+\\
&8x^3uv^5w+xu^4v^4w^4+6xuv^5w^3-36u^2v^4w^2+\\
&12u^2v^4w^4-4u^3v^3w^5-2x^4u^3v^5w+x^3u^6v^2w^2-\\
&6x^3u^5v^3w-192x^2v^4w^2-48u^2v^4-12x^3u^2v^6-\\
&192x^2u^4w^2+8x^2u^5vw+192x^2u^2w^2-48u^4v^2+\\
&144xu^3v^3w-2x^4u^5v^3w-192x^2uvw^3+176x^2uv^2+\\
&24x^3u^4v^4+176x^2u^2v^4-160x^2u^3v^3w-6x^3u^3v^5w+\\
&6x^2u^5vw^3-12u^5vw^3+x^3u^2v^6w^2-2x^2u^5v^3w^3+\\
&256x^5)
\end{aligned}
$$

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