Lecture 2-13-1
Polynomial systems, computer algebra and applications

Polynomials, Solution sets, Gröbner bases

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First informations

Pedagogical team.

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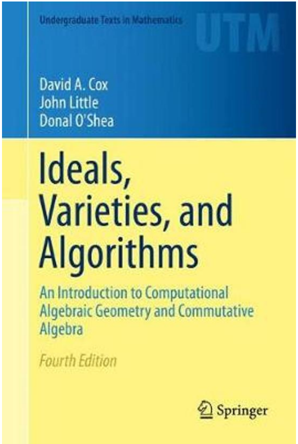
All slides and companion lecture notes (including exercises) will be available at

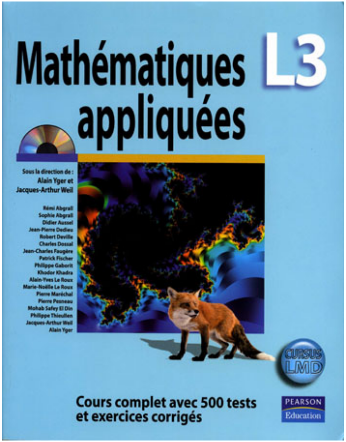
https://www-polsys.lip6.fr/∼jcf/Teaching/index.html

The course is taught in English upon explicit request.

Research internships / PhD positions available on the web. **Contact the teachers asap** if you get interested.

Textbooks





Mathematical background (mainly commutative algebra and some tapas of algebraic geometry) is introduced when needed.

This course is research oriented: it includes new recently published results.

Introduction

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Provided $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n)$ and variables $x_1, \dots, x_n$,

$\boldsymbol{\alpha}$ encodes $\boldsymbol{x}^{\boldsymbol{\alpha}} = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$.

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The total degree of $\boldsymbol{\alpha}$ is the sum $\alpha_1 + \cdots + \alpha_n$.

Univariate monomials ($n = 1$) are naturally ordered by their degree and this order is compatible with multiplication

$$(m_1 \prec m_2 \Rightarrow \forall m \in \mathbb{N}^n, m \times m_1 \prec m \times m_2).$$

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We will define and use such orders which are called admissible orders.

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A polynomial is an element of this $\mathbb{K}$-vector space E. All in all, it is represented as a finite sequence of elements $(c_{\boldsymbol{\alpha}_1}, \ldots, c_{\boldsymbol{\alpha}_\ell})$ of $\mathbb{K}$, indexed by finitely many elements of $\mathbb{N}^n$. Mathematically, it is a finite linear combination of monomials over $\mathbb{K}$.

Equipped with variables $x_1, \ldots, x_n \quad \rightsquigarrow c_{\boldsymbol{\alpha}_1} x_1^{\alpha_{1,1}} \cdots x_n^{\alpha_{1,n}} + \cdots + c_{\boldsymbol{\alpha}_\ell} x_1^{\alpha_{\ell,1}} \cdots x_n^{\alpha_{\ell,n}}$.

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The set of polynomials with variables $x_1, \ldots, x_n$ with base field $\mathbb{K}$ is denoted by $\mathbb{K}[x_1, \ldots, x_n]$.

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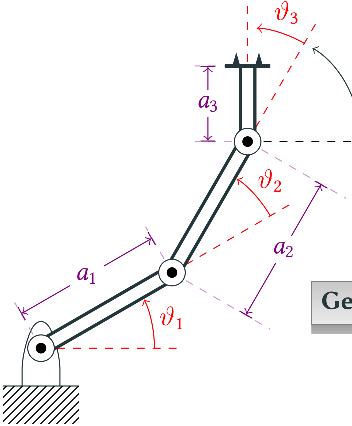
Arithmetic size. A polynomial $f \in \mathbb{K}[x_1, \ldots, x_n]$ of degree D is encoded with an array of coefficients of length $\binom{n+D}{n}$.

Polynomial systems in engineering sciences (I)



$$\begin{cases} x_{\text{end}} &= a_1\cos(\vartheta_1) + a_2\cos(\vartheta_1+\vartheta_2) + \\ & \quad a_3\cos(\vartheta_1+\vartheta_2+\vartheta_3) \\ y_{\text{end}} &= a_1\sin(\vartheta_1) + a_2\sin(\vartheta_1+\vartheta_2) + \\ & \quad a_3\sin(\vartheta_1+\vartheta_2+\vartheta_3) \\ \psi &= \vartheta_1+\vartheta_2+\vartheta_3 \end{cases}$$

Geometric properties → **kinematic map**

$$\mathscr{K} : SO(2)^n \to E.$$

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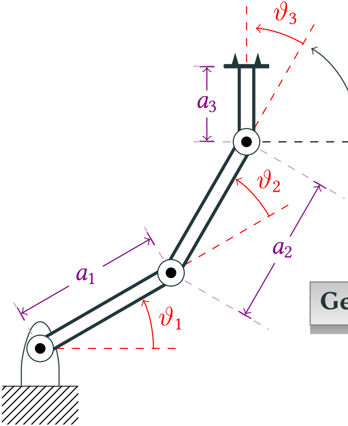

$$
\begin{cases}
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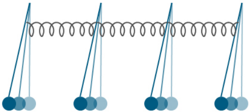
Polynomial systems are ubiquitous in robotics, mechanics, and some areas of biology and chemistry.

Polynomial systems in engineering sciences (II)



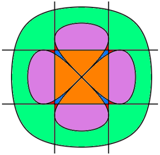
A dynamical system of 4 coupled oscillators: maximal number of equilibria?

$\rightsquigarrow$ Kuramoto model of 4 oscillators

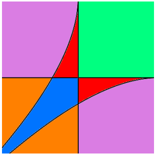
Maximal number of real solutions of

$$\begin{cases} x_{2i-1}^2 + x_{2i}^2 = 1, x_7 = 0, x_8 = 1 \\ \sum_{j=1}^{4} (x_{2i-1}x_{2j} - x_{2i}x_{2j-1}) = y_i, i = 1 \ldots 3 \end{cases}$$

[Xin, Kikkawa, Liu '16] conjectured at most **10** real solutions



Section $y_1 + y_2 + 2y_3 = 0$



Zoom in: **2**, **4**, **6**, **8**, **10**

Definitive answer through polynomial system solvers

Polynomial systems solving in mathematics

Algebraic methods for polynomial system solving and more generally, computer algebra can be used to prove some mathematical results (or disprove mathematical conjectures).

Theorem. Surfaces of degree 3 always contain lines and conics.

Noether–Lefschetz theorem $\Longrightarrow$ surfaces of degree ≥ 4 almost never do.

What about some special surfaces of degree 4? $\cos(t)f + \sin(t)g = 0$

Using the `msolve` library

SORBONNE UNIVERSITÉ

TECHNISCHE UNIVERSITÄT KAISERSLAUTERN

UNIVERSITY OF OREGON

Polynomial systems in cryptography

Context. A (Alice) wants to send a private message to B (Bob).

Let $\mathbb{K}$ be a finite field and $R = \mathbb{K}[x_1, \ldots, x_n]$.

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To send a message $m \in \mathbb{K} \in \mathbb{K}^n$, A picks a matrix M in $R^{n\times s}$ and sends

$$c = m + M \,.\, \boldsymbol{f} \in R^n$$

Decoding amounts to substitute variables in c by ξ.

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National Institute of Standards and Technology
U.S. Department of Commerce

Multivariate cryptography and post-quantum cryptography are hot topics

Features of polynomial systems of equations

Linear systems of equations may have

infinitely many solutions or 0 or 1 solution.

Gaussian elimination $\Longrightarrow$ Triangular rewriting $\Longrightarrow$ Solving

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Non-linear polynomial systems of equations may have

infinitely many solutions $\qquad x_1^2 + x_2^2 - 1 = 0$

0 solution $\qquad x_1 x_2 - 1 = x_1 = 0$

Finitely many solutions $\qquad x_1^2 + x_2^2 - 1 = x_1 - x_2 = 0$

They can have really **a lot of solutions** $\qquad x_1^2 - 2 = \cdots = x_n^2 - 2 = 0$

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... and their coordinates may lie outside the field generated by the input coefficients.

What does “solving” mean?

Depends a lot on the application and on the base field $\mathbb{K}$

- $\mathbb{K} = \mathbb{Q} \rightsquigarrow$ extract informations on real or complex solutions
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Algebraic closure – definition

Let $\mathbb{K}$ be a field. An algebraic closure of $\mathbb{K}$, is a field $\overline{\mathbb{K}}$, containing $\mathbb{K}$ s.t. all univariate polynomials of degree d with coefficients in $\overline{\mathbb{K}}$ have d solutions counted with multiplicities.

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Algebraic methods $\rightarrow$ Triangular rewriting $\rightarrow$ "Solving" over $\overline{\mathbb{K}}$

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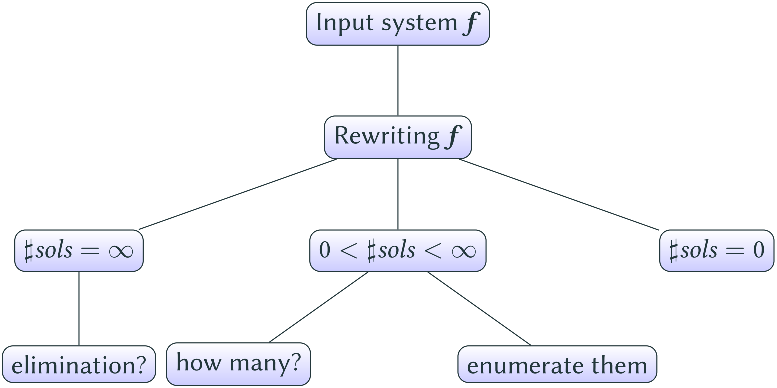

Towards Gröbner bases

Let $\mathbb{K}$ be a field and $R = \mathbb{K}[x_1, \ldots, x_n]$.

	Linear systems	**Polynomial systems**
Equations	$\ell_1 = \cdots = \ell_s = 0$	$f_1 = \cdots = f_s = 0$
Algebraic object	Vector space $V = \{\sum_i a_i \ell_i \mid a_i \in \mathbb{K}\}$	Ideal generated by the f_i's $I = \{\sum_i q_i f_i \mid q_i \in R\}$
Algorithm	Gaussian elimination (variable ordering)	"Elimination of monomials" (monomial ordering $\rightsquigarrow$ term rewriting)
Output	Triangular basis of V	**Gröbner basis of I**

Gröbner bases and polynomial system solving

Let $\mathbb{K}$ be a field, $R = \mathbb{K}[x_1, \ldots, x_n]$, $\prec$ a monomial ordering and

$$\boldsymbol{f} = (f_1, \ldots, f_s) \subset R.$$

Provided $\prec$, a Gröbner basis G of the ideal generated by $\boldsymbol{f}$ provides a canonical description of it.

Emptiness decision

A non-zero constant $a \in \mathbb{K}$ lies in G.

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- Unique representative of f modulo the ideal (provided $\succ$)
- Allows to compute “modulo” the equations

$\rightsquigarrow$ description of the solutions

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One can choose $\prec$ such that G has the following shape

$$G = \begin{cases} T_n \subset \mathbb{K}[x_1, \ldots, x_n] \\ \qquad\quad \vdots \\ T_2 \subset \mathbb{K}[x_{n-1}, x_n] \\ T_1 \subset \mathbb{K}[x_n] \end{cases}$$

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Shape position

When $s \geq n$, and when the solution set in $\overline{\mathbb{K}}^n$ is finite, G has (most of the time) the following so-called shape position

$$w, x_2 - v_2, \ldots, x_n - v_n \text{ with } w, v_i \in \mathbb{K}[x_1]$$

- Note that, in this case, solutions in $\mathbb{K}^n$ can be recovered

Julia Demo

Basic notions of algebra and geometry

Algebraic sets and ideals

Ideals and solution sets

Let $\mathbb{K}$ be a field, $\overline{\mathbb{K}}$ an algebraic closure of $\mathbb{K}$, and $R = \mathbb{K}[x_1, \ldots, x_n]$.

Ideal

An ideal I of R is a non-empty subset of R such that for all f, g in I and $h \in R$, $f + g \in I$ and $hf \in I$.

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Lemma – Definition

Let $S \subset R$. Then

$$\left\{ \sum_{i=1}^{s} q_i f_i \mid q_i \in R, (f_1, \ldots, f_s) \subset S \right\}$$

is an ideal of R. It is called the ideal generated by S, denoted by $\langle S \rangle$.

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Remark. Let $\xi \in \overline{\mathbb{K}}^n$ such that $f_i(\xi) = 0$ for $1 \le i \le s$.

Then for all $g \in \langle \boldsymbol{f} \rangle$, $g(\xi) = 0$.

Ideals and solution sets

Let $\mathbb{K}$ be a field, $\overline{\mathbb{K}}$ an algebraic closure of $\mathbb{K}$, and $R = \mathbb{K}[x_1, \ldots, x_n]$.

Definition

A $\mathbb{K}$-algebraic set V (also called algebraic variety) of $\overline{\mathbb{K}}^n$ is a subset of $\overline{\mathbb{K}}^n$ such that there exists a subset $S \subset R$ such that

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- Let I and $\mathcal{J}$ be ideals of R. It holds that $I \subset \mathcal{J}$ iff $V(\mathcal{J}) \subset V(I)$.
- Let I and $\mathcal{J}$ be ideals of R. It holds that $V(I \cap \mathcal{J}) = V(I) \cup V(\mathcal{J})$.

Hilbert's basis theorem

Let $\mathbb{K}$ be a field, $\overline{\mathbb{K}}$ an algebraic closure of $\mathbb{K}$, and $R = \mathbb{K}[x_1, \ldots, x_n]$.

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Algebraic methods $\rightsquigarrow$ **Appropriate bases of ideals**

Noetherianity

Let $\mathbb{K}$ be a field, $\overline{\mathbb{K}}$ an algebraic closure of $\mathbb{K}$, and $R = \mathbb{K}[x_1, \ldots, x_n]$.

Ascending chains of ideals

Let $(I_i)_{i \in \mathbb{N}}$ be a sequence of ideals such that

$$I_1 \subset I_2 \subset \cdots \subset I_i \subset I_{i+1} \subset \cdots$$

There exists $k \in \mathbb{N}$ such that for all $\ell \geq k$, $I_k = I_\ell$.

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- **Example.** Take $f_1 = x_1 x_2 - 1$ and $f_2 = x_1$. Then $1 \in \langle f_1, f_2 \rangle$

  (since $1 = x_2 f_2 - f_1$)

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- Warning: applies to $\overline{\mathbb{K}}$ only.

  Example. Take $\mathbb{K} = \mathbb{R}$ and $I = \langle x_1^2 + x_2^2 + 1 \rangle$.

Ideals associated to sets, Hilbert's Nullstellensatz

Let $\mathbb{K}$ be a field, $\overline{\mathbb{K}}$ an algebraic closure of $\mathbb{K}$, and $R = \mathbb{K}[x_1, \ldots, x_n]$.

Definition – Lemma

Let $E \subset \overline{\mathbb{K}}^n$. Consider the subset S of R associated to E defined as

$$S = \{f \in R \mid \forall \xi \in E, \quad f(\xi) = 0\} .$$

It holds that S is an ideal, which is called ideal associated to E and denoted by $\mathrm{Id}(E)$.

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- **Exercise.** Prove that $\mathrm{Id}(\mathbb{R}) = \langle 0 \rangle$.

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$$x_1 \in \mathsf{Id}(V(I)) \qquad \text{and} \qquad x_1 \notin I.$$

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Let $f \in R$ and $I \subset R$ be an ideal. If $f \in \mathrm{Id}(V(I))$, then there exists $k \in \mathbb{N}$ such that $f^k \in I$.

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Let $f \in R$ and $I \subset R$ be an ideal. If $f \in \mathrm{Id}(V(I))$, then there exists $k \in \mathbb{N}$ such that $f^k \in I$.

Definition/Lemma – radical ideal

Let $I \subset R$ be an ideal. The set

$$\{f \in R \mid \exists k \in \mathbb{N} \text{ such that } f^k \in I\}$$

is an ideal. It is called the radical of I and denoted by $\sqrt{I}$.

Ideal – Variety correspondence

Let $\mathbb{K}$ be a field, $\overline{\mathbb{K}}$ an algebraic closure of $\mathbb{K}$, and $R = \mathbb{K}[x_1, \ldots, x_n]$.

- Let $I \subset R$ be an ideal. It holds that

$$\sqrt{I} = \mathsf{Id}(V(I)).$$

- Let $V \subset \overline{\mathbb{K}}^n$ be an algebraic set. It holds that

$$V(\mathsf{Id}(V)) = V.$$

Zariski topology

This slide anticipates our future study of the complexity of Gröbner bases computations (under so-called regularity assumption).

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Let $\mathbb{K}$ be a field, $\overline{\mathbb{K}}$ an algebraic closure of $\mathbb{K}$ and $n \in \mathbb{N} - \{0\}$

One can equip $\overline{\mathbb{K}}^n$ with a so-called Zariski topology, where the class of closed sets is the class of algebraic sets in $\overline{\mathbb{K}}^n$.
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- The Zariski closure of a subset W in $\overline{\mathbb{K}}^n$ is the smallest (for the partial order induced by inclusion) algebraic set which contains W.
- The Zariski topology is less fine that the Euclidean topology.
  Example. The open (for the Euclidean topology) disk centered at 0 of radius 1 in $\mathbb{C}$ is not an open set for the Zariski topology. Its Zariski closure is $\mathbb{C}$.

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- The set of polynomials of degree $\leq D$ with coefficients in $\overline{\mathbb{K}}$ is a finite dimensional vector space. It is isomorphic to $\overline{\mathbb{K}}^N$ with $N = \binom{n+D}{D}$. A property $\mathscr{P}$ on polynomials is generic iff there exists a non-empty Zariski open subset $U \subset \overline{\mathbb{K}}^N$ such that $\mathscr{P}$ holds for any $f \in U$.

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Dimension

Let $V \subset \overline{\mathbb{K}}^n$ be an algebraic set. The dimension of V is the maximum integer d such that there exists $\boldsymbol{\iota} = \{i_1, \ldots, i_d\} \subset \{1, \ldots, n\}$ such that $\pi_{\boldsymbol{\iota}}(V)$ contains a non-empty Zariski open subset of $\overline{\mathbb{K}}^d$.

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- By convention, the dimension of the empty set it -1.
- When V is a finite set of points, it has dimension 0.
- A hypersurface (defined by a single polynomial) has dimension $n-1$.

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Other example.

$$(x_1^2 + x_2^2)(x_1 - 1) = (x_1^2 + x_2^2)(x_2 - 1) = 0$$

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Let $V \subset \overline{\mathbb{K}}^n$ be a non-empty algebraic set of dimension d.

Lemma

For a generic choice of a $(n-d)$-dimensional affine linear subspace $\mathscr{L}_{n-d}$, $V \cap \mathscr{L}_{n-d}$ has dimension 0.

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Lemma – Definition

There exists $\delta \in \mathbb{N}$ such that, for a generic choice of a $(n-d)$-dimensional affine linear subspace $\mathscr{L}_{n-d}$, $V \cap \mathscr{L}_{n-d}$ has dimension 0 and cardinality δ. We call δ the degree of V and we denote it by $\deg(V)$.

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- By convention, the degree of the empty set is 0.

Bézout's theorem

Bézout theorem

Let V and W be two algebraic sets in $\overline{\mathbb{K}}^n$. Then

$$\deg(V \cap W) \leq \deg(V)\deg(W)$$

Gröbner bases

Definitions and first properties

Reminder of motivation

$\boxed{\text{Ideal membership problem}}$

Hilbert's weak Nullstellensatz.

Rewriting input polynomial systems

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Rewriting input polynomial systems

Picking $x_1 \succ x_2$

$f_1 = \boxed{x_1} + x_2 - 1$

$f_2 = \boxed{x_1} - x_2 + 1$

$\boxed{f_1 - f_2} = 2x_2 - 2$

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What about ?

$f_1 = x_1^2 + x_2^2 + x_1x_2 - 1$

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We need more ingredients

Monomial orderings

Admissible monomial orderings

Let $\prec$ be a total order over $\mathbb{N}^n$. We say that $\prec$ is an admissible monomial ordering if the following holds:

- $\mathbf{0} \preceq \boldsymbol{\alpha}$ for all $\boldsymbol{\alpha} \in \mathbb{N}^n$;
- if $\boldsymbol{\alpha} \prec \boldsymbol{\beta}$ then for any $\boldsymbol{\gamma} \in \mathbb{N}^n$ it holds that $\boldsymbol{\alpha} + \boldsymbol{\gamma} \prec \boldsymbol{\beta} + \boldsymbol{\gamma}$

  $\prec$ is compatible with multiplication
- there is no infinitely decreasing sequence $(\boldsymbol{\alpha}_i)_{i \in \mathbb{N}}$

There are many different ways to define admissible monomial orderings, which may have additional properties.

Some examples (I)

Lexicographical monomial ordering

Let $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$ and $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_n)$ in $\mathbb{N}^n$.

$$\boldsymbol{\alpha} \prec_{lex} \boldsymbol{\beta} \iff \exists i \text{ such that } \begin{cases} \alpha_j = \beta_j \text{ for } j < i \\ \alpha_i < \beta_i \end{cases}$$

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This ordering eliminates variables at first. It compares first the exponent of x_1, in case of equality it compares the exponent of x_2, and so on.

Examples

- $x_3^{10} \prec_{lex} x_2^3 \prec_{lex} x_1$
- $1 \prec_{lex} x_2 \prec_{lex} x_2^2 \prec_{lex} x_2^{1000} \prec_{lex} x_1 \prec_{lex} x_1 x_2 \prec_{lex} x_1^2$

Some examples (II)

Graded lexicographical monomial ordering

Let $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$ and $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_n)$ in $\mathbb{N}^n$.

$$\boldsymbol{\alpha} \prec_{grlex} \boldsymbol{\beta} \iff \sum_{i=1}^{n} \alpha_i < \sum_{i=1}^{n} \beta_i \text{ or } \begin{cases} \sum_{i=1}^{n} \alpha_i = \sum_{i=1}^{n} \beta_i \\ \text{ and there exists } i \text{ such that} \\ \alpha_j = \beta_j \text{ for } j < i \\ \alpha_i < \beta_i \end{cases}$$

This ordering first filters monomials w.r.t. their degrees and next applies the lexicographical ordering.

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This ordering first filters monomials w.r.t. their degrees and next applies the lexicographical ordering.

Feature. All monomials are preceded by a finite number of other monomials.

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Graded reverse lexicographical monomial ordering

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Block orderings

Let $\prec_1$ and $\prec_2$ be two admissible monomial orderings over $\mathbb{N}^i$ and $\mathbb{N}^j$ with $n = i + j$.

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Some more comments on monomial orderings

- The lexicographical ordering is the one which will enable triangular rewritings of the input system.
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> The *grevlex* ordering is better suited to computing Gröbner bases as it is related to some intrinsic complexity measures for polynomial ideals (notions of regularity that will appear later in the course).

Bayer/Stillmann'87

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Change of ordering algorithms?

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- The leading coefficient of f w.r.t. $\prec$, denoted by $\mathrm{LC}_{\prec}(f)$, is $c_{\boldsymbol{\alpha}_i}$.

Leading monomials, coefficients and terms (II)

Consider again

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Observe that $g_1 \in \langle f_1, f_2 \rangle$ brings a new information:
$\mathsf{LM}_{grevlex}(g_1) \notin \langle \mathsf{LM}_{grevlex}(f_1), \mathsf{LM}_{grevlex}(f_2) \rangle$
What could be the next steps?

Eliminating terms – towards critical pairs

$$\begin{aligned} f_1 &= \boxed{x_1^2} + 2x_2^2 + 5x_1x_2 - 1 \\ f_2 &= \boxed{x_1^2} - 3x_2^2 - 2x_1x_2 + 1 \end{aligned} \rightsquigarrow \begin{aligned} f_1 &= \boxed{x_1^2} + 2x_2^2 + 5x_1x_2 - 1 \\ f_2 &= \boxed{x_1^2} - 3x_2^2 - 2x_1x_2 + 1 \\ g_1 = f_1 - f_2 &= \boxed{7x_1x_2} + 5x_2^2 \end{aligned}$$

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$\rightsquigarrow$ $g_2 - g_3$ brings a new important one.

Remember the ascending chains of ideals theorem...

Summary

Admissible monomial orders

Mimic degree extension step in Euclidean divison

New specific question: ideal membership for monomial ideals

More is needed...

S-polynomials (I)

Let $\mathbb{K}$ be a field, $R = \mathbb{K}[x_1, \ldots, x_n]$ and $\prec$ an admissible monomial ordering.

Let $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$ and $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_n)$ be two monomials of R. The least common multiple of $\boldsymbol{\alpha}, \boldsymbol{\beta}$ ($\mathrm{lcm}(\boldsymbol{\alpha}, \boldsymbol{\beta})$) is the monomial $(\max(\alpha_1, \beta_1), \ldots, \max(\alpha_n, \beta_n))$.

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- Back to a notation with variables, it generates $\langle \boldsymbol{x}^{\alpha} \rangle \cap \langle \boldsymbol{x}^{\boldsymbol{\beta}} \rangle$.
- For f, g in $R - \{0\}$, we define $\mathsf{lcm}_{\prec}(f, g) = \mathsf{lcm}(\mathsf{LM}_{\prec}(f), \mathsf{LM}_{\prec}(g))$.

S-polynomials (II)

Let f and g be in $R - \{0\}$. Let $\lambda = \mathsf{lcm}_{\prec}(f, g)$.
We define the S-polynomial of (f, g) w.r.t. $\prec$ as

$$\mathsf{spol}_{\prec}(f, g) = \frac{\lambda}{\mathsf{LT}_{\prec}(f)} f - \frac{\lambda}{\mathsf{LT}_{\prec}(g)} g$$

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- As illustrated in the previous example, S-polynomials play a prominent role in discovering new relevant polynomials g in some polynomial ideal $\langle f_1, \ldots, f_s \rangle$.

$$\mathsf{LM}_{\prec}(g) \notin \langle \mathsf{LM}_{\prec}(f_1), \ldots, \mathsf{LM}_{\prec}(f_s) \rangle$$

Gröbner bases – Definition

Let $\mathbb{K}$ be a field, $R = \mathbb{K}[x_1, \ldots, x_n]$ and $\prec$ an admissible monomial ordering over R.

Definition

Let $I \subset R$ be an ideal. One says that $G \subset R$ is a Gröbner basis for $(I, \prec)$ if the following conditions hold:

- G is finite;
- $G \subset I$;
- $\langle \mathsf{LM}_\prec(g) \mid g \in G \rangle = \langle \mathsf{LM}_\prec(f) \mid f \in I \rangle$.

Ideal membership problem for monomial ideals

Let $\mathbb{K}$ be a field and $R = \mathbb{K}[x_1, \ldots, x_n]$.

Monomial ideals

Let $I \subset R$ be an ideal. One says that I is a monomial ideal iff there exists a subset S of monomials such that $I = \langle S \rangle$.

Note that we do not assume S to be finite. Hilbert's basis theorem implies that I is finitely generated by elements of R, not by monomials.

Lemma

Let $I \subset R$ be a monomial ideal and $S \subset R$ be a set of monomial generators for I. Let $\boldsymbol{x}^{\boldsymbol{\alpha}}$ be a monomial. The following holds:

$$\boldsymbol{x}^{\boldsymbol{\alpha}} \in I \iff \boldsymbol{x}^{\boldsymbol{\alpha}} \text{ is divisible by some monomial in } S$$

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Dickson's Lemma

Let $I \subset R$ be a monomial ideal. It holds that I has a finite monomial basis.

Hilbert series (I)

Let $I = \langle \boldsymbol{x}^{\boldsymbol{\alpha}_1}, \ldots, \boldsymbol{x}^{\boldsymbol{\alpha}_s} \rangle$

We define the Hilbert function as follows:

$$d \mapsto \mathsf{HF}_I(d) = \sharp\{\boldsymbol{\beta} \in \mathbb{N}^n \mid \deg(\boldsymbol{x}^{\boldsymbol{\beta}}) = d \text{ and } \boldsymbol{x}^{\boldsymbol{\beta}} \notin I\}.$$

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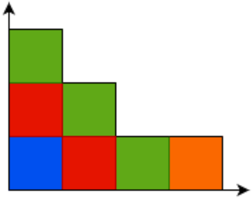

Take $I = \langle x_1^4, x_1^2 x_2, x_1 x_2^2, x_2^3 \rangle$

$$\mathrm{HS}_I(t) = 1 + 2t + 3t^2 + t^3$$