Computing Roadmaps in Smooth Real Algebraic Sets

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Abstract

Let $(f_1, \dots, f_s)$ be polynomials in $\mathbb{Q}[X_1, \dots, X_n]$ of degree bounded by D that generate a radical equidimensional ideal of dimension d and let $\mathcal{V} \subset \mathbb{C}^n$ be the locus of their complex zero set which is supposed to be smooth. A *roadmap* in $\mathcal{V} \cap \mathbb{R}^n$ is a real algebraic curve contained in $\mathcal{V} \cap \mathbb{R}^n$ which has a non-empty and connected intersection with each connected component of $\mathcal{V} \cap \mathbb{R}^n$.

The classical strategy to compute roadmaps, which is due to J. Canny, is based on computing a polar variety of dimension 1 and a recursion on the studied variety intersected with fibers taken above a critical value of a projection. Thus, it requires computations with real algebraic numbers and introduces singularities at each recursive call. We show how to slightly modify this strategy in order to avoid the use of real algebraic numbers and to deal with smooth algebraic sets at each recursive call in the case where the input variety is smooth. Our complexity is $h^d D^{\mathcal{O}(n)}$ operations in $\mathbb{Q}$ where h bounds the number of recursive call in our algorithm. This quantity is related to the geometry of $\mathcal{V} \cap \mathbb{R}^n$ and is bounded by $D^{\mathcal{O}(n)}$. We briefly report on some experiments done with a preliminary implementation of our algorithm.

Keywords. Polynomial System Solving, Real Solutions, Connectedness, Complexity.

Introduction

Let $(f_1, \dots, f_s)$ be polynomials in $\mathbb{Q}[X_1, \dots, X_n]$ of degree bounded by D that generate a radical equidimensional ideal of dimension d. Let $\mathcal{V} \subset \mathbb{C}^n$ be the algebraic set defined by $f_1 = \cdots = f_s = 0$ which is supposed to be smooth in the sequel. A *roadmap* $\mathcal{R}$ associated to $\mathcal{V} \cap \mathbb{R}^n$ is an algebraic curve contained in $\mathcal{V}$ having a *non-empty* and *connected* intersection with each connected component of $\mathcal{V} \cap \mathbb{R}^n$. This paper is devoted to design an efficient algorithm computing roadmaps in smooth real algebraic sets.

Computing roadmaps allows to reduce general connectivity decision problems to connectivity decision in dimension 1 for which there exist algorithms (see [4]). The problem of deciding connectivity is motivated by problems arising in robot motion planning where deciding if two given points belong to the same connected component of a semi-algebraic set is a question of first importance (see [17]). In this context, the computation of a *roadmap* can be seen as a preliminary step (see [3]) using connecting subroutines between these points and the computed roadmap. Roadmaps can also be used to compute the number of connected components of a semi-algebraic set and parametrized versions of some roadmap

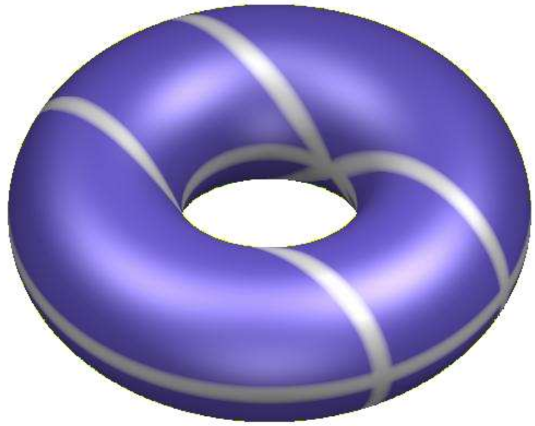

Figure 1: Canny's roadmap in the case of a torus.

algorithms are used to obtain a semi-algebraic description of the connected components of a semi-algebraic set (see [4]).

Roadmaps can be extracted from a cylindrical algebraic decomposition but such an approach leads to algorithmic solutions which are doubly exponential in the number of variables. The notion of roadmap is explicitly introduced in 1987 by J. Canny (see [5, 6]) who provides an algorithm computing roadmaps for semi-algebraic set in $k^n D^{\mathcal{O}(n^4)}$ operations in $\mathbb{Q}$ (where k is the number of inequalities defining the considered semi-algebraic set. In the case of real algebraic sets we are considering here, the complexity of his algorithm is $D^{\mathcal{O}(n^4)}$. A Monte-Carlo version of Canny's algorithm computes roadmaps in $D^{\mathcal{O}(n^2)}$. Further developments can be found in [] and in particular [3] where an algorithm computing roadmaps in a semi-algebraic set having a complexity $k^{d'} D^{\mathcal{O}(n^2)}$ where k is the number of inequalities and d' the dimension of the considered semi-algebraic set. In the case of real algebraic sets, the latter algorithm has a complexity $D^{\mathcal{O}(n^2)}$ arithmetic operations in $\mathbb{Q}$. All the algorithms cited above are, in the case of real algebraic sets, based on Canny's strategy we present below.

Canny's strategy to compute roadmaps. Suppose that $\mathcal{V} \cap \mathbb{R}^n$ is compact. Canny's algorithm computes a "silhouette", i.e. the critical locus of a projection on a plane P restricted to $\mathcal{V}$. Since $\mathcal{V} \cap \mathbb{R}^n$ is compact, this silhouette has a non-empty intersection with each connected component of $\mathcal{V} \cap \mathbb{R}^n$ but this intersection may be non connected. In order to connect the components of the silhouette, classical results of Morse theory show that it is sufficient to construct roadmaps in the slices which are fibers taken above the critical values of the projection on a line lying in P. Figure 1 illustrates this process in the case of a torus in $\mathbb{R}^3$.

The construction of these roadmaps is done by considering a 1-dimensional silhouette in the slice and once again the critical values of some projection on a line restricted to the silhouette and computing a slice in the slice, and so on until one has constructed a 1-dimensional roadmap. Thus the above construction is based on the recursive calls to a procedure computing:

- a critical locus of the projection on the plane (X_1, X_2);
- the set E of critical values of a projection on X_1

on the set of polynomials defining $\mathcal{V}$ where X_1 is replaced by v for each v in E and the set of variables $(X_1, \ldots, X_n)$ is obviously replaced by $(X_2, \ldots, X_n)$ so that the next silhouette is computed relatively to the plane (X_2, X_3).
Remark that each slice defines a singular variety and is defined above a real algebraic number. Thus, deformation techniques based on the introduction of infinitesimals are required to deal with these singular varieties. This, and the use of real algebraic numbers, makes the arithmetic on which the computations are performed extremely heavy. Thus, in despite to the rather good complexity of roadmap algorithms, they had never been implemented. In this paper, we show how to slightly modify the above geometric procedure to avoid the aforementioned problems in the case where the input polynomials define a smooth algebraic variety.

Slight modification of Canny's strategy in smooth situations. As above, suppose $\mathcal{V} \cap \mathbb{R}^n$ is compact. We do the following assumption:

- (H_1) the critical locus C of the projection on (X_1, X_2) restricted to $\mathcal{V}$ is 1-dimensional and is smooth.
- (H_2) Above each critical value of the projection on X_1 restricted to C, there exists a unique critical point.

In this case, one can connect the connected components of C lying in the same connected components of $\mathcal{V} \cap \mathbb{R}^n$ by considering fibers *between* each critical value of the projection on X_1 restricted to C. Figure 2 illustrates this process.

Thus, our algorithm is based on the recursive call of the following procedure:

- compute the critical locus C of the projection on the plane (X_1, X_2);
- compute a set $E \subset \mathbb{Q}$ containing a rational number between each critical value of the projection on X_1 restricted to C.

on the set of input polynomials defining $\mathcal{V}$ where X_1 is replaced by v for each $v \in E$.

Remark now that since the fibers are taken above a regular value of the projection on X_1, the slices are smooth algebraic varieties. Moreover, the computations are no more performed using real algebraic numbers since one can take the fibers above rational numbers.

The proof of the connectedness of the roadmap constructed with respect to the above scheme closely follows the one of Canny's algorithm.

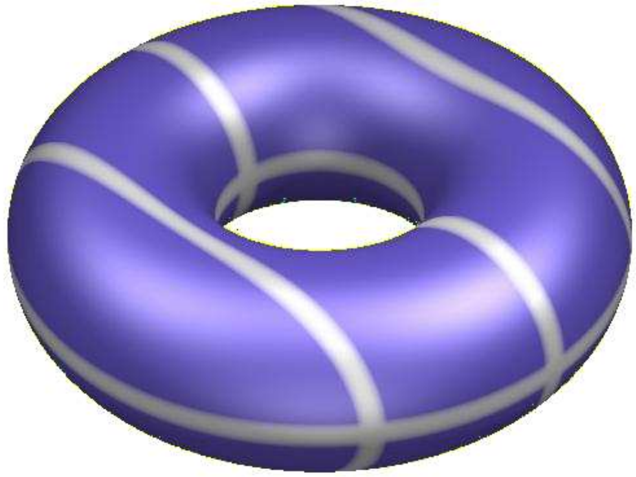

Figure 2: Roadmap obtained by modifying Canny's strategy in the case of a torus.

Plan of the paper. In the next section, we show how to reduce the study of smooth real algebraic sets to the study of compact smooth real algebraic sets and that the assumption which has been done in the modification of Canny's strategy is *generically* satisfied. The following section is devoted to the formal description of the algorithm we obtain. In the last section, we study the complexity of our contribution using Lecerf's results (see []) on Geometric Resolution and present some results obtained with our implementation which is based on Gröbner bases computations.

1 Preliminaries

This section is devoted to some preliminaries required in the sequel. We consider here a set of polynomials $(f_1, \ldots, f_s)$ in $\mathbb{Q}[X_1, \ldots, X_n]$ generating a radical equidimensional ideal of dimension d and their complex zero set $\mathcal{V} \subset \mathbb{C}^n$ which is supposed o be smooth.

We first show how to reduce the problem of computing a roadmap in a smooth real algebraic set to computing a roadmap in a smooth and *compact* real algebraic set.

Theorem 1.1. *Let $A = (a_1, \ldots, a_n) \in \mathbb{Q}^n$ and E_A be the set of critical values of the square of the euclidean distance restricted to $\mathcal{V}$. Let R be a rational number which is greater that* $\max(e, e \in E)$ *and X_{n+1} be a new variable. Consider the algebraic variety $\mathcal{W}_A$ defined by:*

$$f_1 = \cdots = f_s = ((X_1 - a_1)^2 + \cdots + (X_n - a_n)^2 + X_{n+1}^2 - R^2)$$

There exists a Zariski-closed subset $\mathcal{A} \subset \mathbb{C}^n$ such that for all $A \in \mathbb{Q}^n \setminus \mathcal{A}$, the following holds:

- *the above polynomial system generates a radical and equidimensional ideal,*

- *$\mathcal{W}_A$ is smooth,*
- *$\mathcal{W}_A \cap \mathbb{R}^{n+1}$ is compact.*

Moreover, the projection of a roadmap computed in $\mathcal{W} \cap \mathbb{R}^{n+1}$ onto $(X_1, \dots, X_n)$ is a roadmap on $\mathcal{V} \cap \mathbb{R}^n$.

Proof. We only provide here a sketch of proof. The existence of the Zariski-closed subset $\mathcal{A}$ such that for all $A = (a_1, \dots, a_n) \in \mathbb{Q}^n \setminus \mathcal{A}$, the system

$$f_1 = \dots = f_s = ((X_1 - a_1)^2 + \dots + (X_n - a_n)^2 + X_{n+1}^2 - R^2)$$

generates a radical equidimensional ideal and defines a smooth algebraic variety comes from its characterization as a set of critical values of a polynomial mapping. We refer to [1] and [2] for similar reasonings.

The compacity of $\mathcal{W}_A$ is obvious. The properties of connectedness of the projection of a roadmap in $\mathcal{W}_A$ are done in [4, Chapter 15]. $\square$

We consider now $\mathbf{A} \in GL_n(\mathbb{Q})$ and, given $f \in \mathbb{Q}[X_1, \dots, X_n]$, we denote by $f^{\mathbf{A}}$ the polynomial $f(\mathbf{A}.X)$ where X denotes the vector $(X_1, \dots, X_n)$. Consider also the canonical projections Π_i:

$$\begin{array}{cccc} \Pi_i : & \mathbb{C}^n & \longrightarrow & \mathbb{C}^i \\ & (x_1, \dots, x_n) & \rightarrow & (x_1, \dots, x_i) \end{array}$$

and denote by $K(\Pi_i, \mathcal{V}^{\mathbf{A}})$ the critical locus of Π_i restricted to $\mathcal{V}^{\mathbf{A}}$ which is defined by:

$$f_1^{\mathbf{A}} = \dots = f_s^{\mathbf{A}} = 0$$

Theorem 1.2. *[2] If $\mathcal{V}$ is smooth and equidimensional, there exists a Zariski-closed subset $\mathcal{A} \subset GL_n(\mathbb{C})$ such that for all $\mathbf{A} \in GL_n(\mathbb{Q}) \setminus \mathcal{A}$, for all $i \in \{1, \dots, d-1\}$, $K(\Pi_i, \mathcal{V}^{\mathbf{A}})$ are smooth equidimensional algebraic varieties.*

From the above result, one deduces easily the following one:

Corollary 1.3. *Up to a generic linear change of coordinates $\mathbf{A} \in GL_n(\mathbb{Q})$, for $j = 1, \dots, d-2$, there exists a Zariski-closed subset $\mathcal{P}_j$ such that for all $(p_1, \dots, p_j) \in \mathbb{Q}^j \setminus \mathcal{P}_j$, $K(\Pi_{j+2}, \mathcal{V}^{\mathbf{A}}) \cap \Pi_j^{-1}(p_1, \dots, p_j)$ is a smooth algebraic curve.*

Remark 1.4. The above corollary implies that at each recursive call, the assumption of smoothness of the 1-dimensional computed critical locus and on the computed fibers are satisfied generically.

Proposition 1.5. *Up to a generic linear change of coordinates $\mathbf{A} \in GL_n(\mathbb{Q})$, for $j = 1, \dots, d-2$, there exists a Zariski-closed subset $\mathcal{P}_j$ such that for all $(p_1, \dots, p_j) \in \mathbb{Q}^j \setminus \mathcal{P}_j$, there is exactly 1 critical point above each critical value of $\pi_{j+2} : (x_1, \dots, x_n) \rightarrow x_{j+2}$ restricted to $K(\Pi_{j+2}, \mathcal{V}^{\mathbf{A}}) \cap \Pi_j^{-1}(p_1, \dots, p_j)$.*

Proof. Applying Corollary 1.3, for all $j = 1, \dots, d-2$, $K(\Pi_{j+2}, \mathcal{V}^{\mathbf{A}}) \cap \Pi_j^{-1}(p_1, \dots, p_j)$ is smooth.

Consider first the case $j = d-2$. This is a consequence of [4, Proposition 7.9] in the case where we consider hypersurfaces. The general case can be obtained by relating the critical points of a generic projection restricted to $\mathcal{V}$ to the ones of the same projection restricted to the hypersurface defined by $f_1^2 + \cdots + f_s^2 - \varepsilon$ where ε is an infinitesimal (see e.g.[5]).

The other cases are obtained by eventually changing $\mathbf{A}$ in such a way that the variables $X_1, \dots, X_{j+1}$ are the only one which are changed so that the previous critical loci of higher dimension are not changed and one recover a generic situation for the other critical loci. □

2 Roadmap Algorithm

We suppose in the sequel that the assumptions (H_1) and (H_2) are satisfied.

The proof of our algorithm is based on the following proposition:

Proposition 2.1. *Let $[a, b] \subset \mathbb{R}$ containing a unique critical value v of Π_1 restricted to $K(\Pi_2, \mathcal{V})$ and $D_{[a,b]}$ be a connected component of $\left(\mathcal{V} \cap \Pi_1^{-1}([a, b])\right) \cap \mathbb{R}^n$. Then, the union $D_{[a,b]} \cap \left(\Pi_1^{-1}(a) \cup K(\Pi_2, \mathcal{V}) \cup \Pi_1^{-1}(b)\right)$ is connected.*

Proof. We provide only a sketch of proof.

Let x and y be two distinct points of $D_{[a,b]} \cap \left(\Pi_1^{-1}(a) \cup K(\Pi_2, \mathcal{V}) \cup \Pi_1^{-1}(b)\right)$. Suppose first $\Pi_1(x) < v < \Pi_1(y)$. since Π_1 realizes a locally trivial fibration on $D_{[a,v[}$ (resp. $D_{]v,b]}$) x (resp. y) can be connected to $D_{[a,b]} \cap \Pi_1^{-1}(a)$ (resp. $D_{[a,b]} \cap \Pi_1^{-1}(b)$ via a continuous path. Let x' and y' be the respective intersection of these continuous path with $\Pi_1^{-1}(a)$ and $\Pi_1^{-1}(b)$. Since $\mathcal{V}$ is smooth, there exists a path $\gamma : t \in [0, 1] \to D_{[a,b]}$ linking x' and y' which does not contain the unique critical point p such that $\Pi_1(p) = v$. We then exhibit a path contained in $K(\Pi_2, \mathcal{V}) \cap D_{[a,b]}$ by studying $\Pi_1^{-1}(\Pi_1(\gamma(t))) \cap K(\Pi_1, \mathcal{V})$, for $t \in [0, 1]$. From the compacity of $\mathcal{V} \cap \mathbb{R}^n$, the smoothness of $\mathcal{V}$ and $K(\Pi_1, \mathcal{V})$ and the assumption (H_2), one deduces a continuous path in $K(\Pi_2, \mathcal{V}) \cap D_{[a,b]}$ which links x' and y'. This configuration is illustrated in Figure 3.

Suppose now that $\Pi_1(x) > v$ and $\Pi_1(y) > v$. If they are both connected to the unique critical point p such that $\Pi_1(p) = v$ by a path in $K(\Pi_2, \mathcal{V}) \cap D_{[a,b]}$ then we are done. If both of them lie on a connected component of $K(\Pi_2, \mathcal{V}) \cap D_{[a,b]}$ which does not contain p, then they can be connected by $D_{[a,b]} \cap \Pi_1^{-1}(b)$ using the properties of locally trivial fibration of Π_2 and the smoothness of $K(\Pi_1, \mathcal{V})$. The same reasoning is done when one of them is connected to p: in this case the connected component of $K(\Pi_2, \mathcal{V})$ containing p has obviously a non-empty intersection with $\Pi_1^{-1}(]v, b])$.

The situation where $\Pi_1(x) < v$ and $\Pi_1(y) < v$ is symmetric to the above one.

At last, one has to deal with the situation where $\Pi_1(x) = v$ and/or $\Pi_1(y) = v$. Since $K(\Pi_1, \mathcal{V})$ is smooth, they are connected to points x' and y' which are in one of the above cases.

□

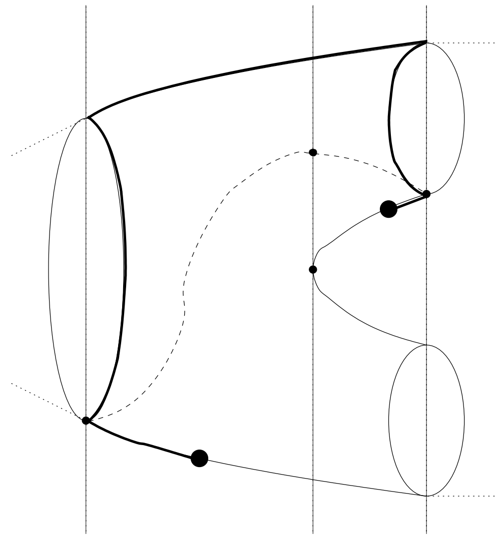

Figure 3: Local connexity.

Algebraic representation of a curve. We focus now on how to represent algebraic curves. Given a polynomial system generating a 1-dimensional ideal, one expects to obtain a parametrization of the curve with coefficients in $\mathbb{Q}(u)$ where u is a parameter.

$$\left\{\begin{array}{lcl} X_n & = & \frac{q_n(u,T)}{q_0(u,T)} \\ & \vdots & \\ X_1 & = & \frac{q_1(u,T)}{q_0(u,T)} \\ q(u,T) & = & 0 \end{array}\right.$$

Such a representation can be valid only outside a finite set of values of the parameter u. Indeed, for almost all specialization e of the parameter u one should retrieve a parametrization of the zero-dimensional set of points which is the intersection of the curve with the hyperplane $u = e$. The variable T here encodes the separating element.

From such a representation, one can compute the number of connected components of the curve (see [4]). From several representations of that kind one can deduce a single one.

Such a parametrization can be computed from a Gröbner basis using [7] or [8] of the input polynomial system and several computations of Rational Univariate Representations (see [14]) using interpolation techniques. Other techniques based on the representation of polynomials by straight-line programs going back to [9, 10, 11] can be used, and more particularly the algorithm of geometric resolution (see [12] and [13]).

Algorithm 2.2. RoadSubRoutine:

- *Input*: A set of polynomials $(f_1, \dots, f_s)$ in $\mathbb{Q}[X_1, \dots, X_n]$ generating an equidimensional radical ideal of dimension d and such that $\mathcal{V}(f_1, \dots, f_s)$ is smooth and compact.

- *Output*: A set $\mathcal{R}$ of parameterizations encoding a silhouette and a set $\mathcal{S} \subset \mathbb{Q}$ of rational numbers.

1. Compute a rational parametrization $\mathcal{R}$ encoding the critical locus of Π_2 restricted to $\mathcal{V}(f_1, \ldots, f_s)$.
2. Compute the critical values of Π_1 restricted to the curve encoded by $\mathcal{R}$.
3. Construct a set $\mathcal{S}$ of rational numbers such that there is exactly one element of $\mathcal{S}$ between each critical value of Π_1 restricted to the curve encoded by $\mathcal{R}$.
4. Return $\mathcal{R}$ and $\mathcal{S}$.

Remark 2.3. The critical locus of Π_2 restricted to $\mathcal{V}(f_1, \ldots, f_s)$ can be obtained by the vanishing of suitable minors of $\mathrm{Jac}(f_1, \ldots, f_s)$ or by using Lagrange's system. In the latter case, one has to compute a rational parametrization of the critical locus after eliminating Lagrange's multipliers.

The critical values of Π_1 restricted to $K(\Pi_2, \mathcal{V}(f_1, \ldots, f_s))$ can be obtained by computing the discriminant of the polynomial $q(u, T)$ in the parametrization $\mathcal{R}$ if the separating element can be chosen equaled to X_1. Since one works with generic coordinates, this is the case.

Algorithm 2.4. CompactRoadMap:

- *Input*: A set of polynomials $(f_1, \ldots, f_s)$ in $\mathbb{Q}[X_1, \ldots, X_n]$ generating an equidimensional radical ideal of dimension d and such that $\mathcal{V}(f_1, \ldots, f_s)$ is smooth and compact.
- *Output*: A set of parameterizations encoding a roadmap in $\mathcal{V}(f_1, \ldots, f_s) \cap \mathbb{R}^n$.

1. $\mathcal{L} := []$; $F := [f_1, \ldots, f_s]$
2. if $n = 2$ return F
3. $\mathcal{R}, \mathcal{S} :=$ `RoadSubRoutine`$([f_1, \ldots, f_s])$
4. $\mathcal{L} := \mathcal{R} \cup \mathcal{L}$
5. $\mathcal{L} := \mathcal{L} \cup (\cup_{s \in \mathcal{S}}$`CompactRoadMap`(`Evaluate`$(X_1 = s, F)))$
6. return $\mathcal{L}$

Remark 2.5. This is an approximative description of the algorithm: specializations of the initial variables should be memorized at each recursive call to recover a description of the roadmap in $\mathbb{R}^n$.

Algorithm 2.6. RoadMap:

- *Input*: A set of polynomials $(f_1, \ldots, f_s)$ in $\mathbb{Q}[X_1, \ldots, X_n]$ generating an equidimensional radical ideal of dimension d and such that $\mathcal{V}(f_1, \ldots, f_s)$ is smooth.

- *Output*: A set of parameterizations encoding a roadmap in $\mathcal{V}(f_1, \ldots, f_s) \cap \mathbb{R}^n$.

1. Compute the critical values of the square of the euclidean distance to a generic point $A = (a_1, \ldots, a_n) \in \mathbb{Q}^n$ restricted to $\mathcal{V}$.

2. Choose a rational R greater than the maximum of these critical values.

3. Return the result provided by `CompactRoadMap` with input $f_1, \ldots, f_s, (X_1 - a_1)^2 + \ldots + (X_n - a_n) + X_{n+1}^2 - R$ in $\mathbb{Q}[X_1, \ldots, X_{n+1}]$ after having performed a randomly chosen linear change of variables $\mathbf{A} \in GL_{n+1}(\mathbb{Q})$ on these polynomials.

Remark 2.7. Note that, obviously, if the input variety is already known to have a compact real counterpart, the only useful step in `RoadMap` is the step of linear change of variables. The step of intersecting the variety with the hyperball $(X_1-a_1)^2+\ldots+(X_n-a_n)+X_{n+1}^2-R$ should be avoided as soon as it is possible since it multiplies by 2 the degree of the studied variety by `CompactRoadMap`.

Proof of correctness of the Algorithm. From Theorem 1.1, it is sufficient to prove that `CompactRoadmap` returns the correct result. The proof of correctness is done by induction on the dimension of the studied variety, following [4]. In the case of 1-dimensional variety, we are done. Consider now the general case. We partitionate the X_1-axis by intervals containing a single critical value of Π_1 restricted to $K(\Pi_2, \mathcal{V})$. Then, we make use of Proposition 2.1 to construct the roadmap by eventually passing through the slices if necessary following the proof of [4, Chapter 15, Lemma 15.8]. Since each slice has a dimension lower than the one of the studied variety, the induction hypothesis can be applied and each recursive call to `CompactRoadmap` returns the correct result. □

3 Complexity Estimates and Implementation

We estimate the complexity of our algorithm in the case where the input polynomial system is a complete intersection, i.e. $s = n - d$. Denote by $\mathtt{C}(n, d, D)$ the cost of the procedure `RoadSubRoutine` and by $\mathtt{H}(n, d, D)$ the number of recursive call to `CompactRoadmap` at step 5 of this procedure and by $\mathtt{T}(n, d, D)$ the total cost of our algorithm where:

- n denotes the number of variables,
- d denotes the dimension of the studied variety,
- D denotes the degree of the input polynomial system.

The cost of our algorithm is:

$$\mathtt{T}(n, d, D) = \mathtt{C}(n, d, D) + \mathtt{H}(n, d, D)\mathtt{T}(n-1, d-1, D)$$

We make use of the results of [16] which shows how to use Lagrange's system in conjunction with Lecerf's results [13] to improve the complexity of computing critical points and [13] which bounds the complexity of computing a lifted curve as a parametrized geometric resolution of 1-dimensional variety defined by polynomials of degree D by $D^{\mathcal{O}(n)}$. One deduces that computing a rational parametrization encoding a curve has a cost which is $D^{\mathcal{O}(n)}$ also. Using Remark 2.3, one deduces that $\mathtt{C}(n,d,D) = D^{\mathcal{O}(n)}$.

Remark now that in worst cases, $H(n,d,D)$ is also bounded by $D^{\mathcal{O}(n)}$. Since at each step, the dimension decreases, one deduces that our algorithm has a complexity within $D^{\mathcal{O}(nd)}$ arithmetic operations in $\mathbb{Q}$ which improves the one of [3]. Nevertheless, note that while the algorithm of [3] is deterministic, ours is probabilistic: it relies on assumptions on the genericity of the initial linear change of variables (nevertheless note that the assumptions (H_1) and (H_2) can be checked using tools such as Gröbner bases) and our complexity analysis relies on the use of an algorithm which is itsself probabilistic. Providing a deterministic algorithm to compute roadmaps in d-dimensional algebraic varieties having a complexity within $D^{\mathcal{O}(nd)}$ arithmetic operations seems to remain an open problem.

Nevertheless, remark that the number of recursive call $H(n,d,D)$ is strongly related to the geometry of $\mathcal{V} \cap \mathbb{R}^n$ and is hopefully often less than the Bézout bound. If h denotes the maximum of $\{H(n-i,d-i,D), i = 0,\ldots,d-1\}$ on an instance, our complexity, expressed in terms of h becomes $h^d D^{\mathcal{O}(n)}$ which is more realistic in regard of the practical behavior of our algorithm.

Moreover, the results of Lecerf allow to precise the complexity constant which is here as an exponent. This will be done in a longer version of a paper presenting this work.

A very preliminary implementation of `CompactRoadmap` has been done in `Maple` using Gröbner bases (`FGb` software, written in `C` by J.C. Faugère) and Rational Univariate Representation (`RS` written in `C` by F. Rouillier) and the interface of these softwares with `Maple`. For the moment the output is a set of 1-dimensional Gröbner bases. Up to our knowledge, it is the first implementation computing roadmaps of real algebraic sets which is not based on Cylindrical Algebraic Decomposition. It already allows to compute roadmaps in smooth algebraic sets lying in $\mathbb{C}^6$ of dimension 5 which seems to be out of the domain reachable by Cylindrical Algebraic Decomposition. Timings obviously show the practical impact of the quantity h. This implementation will be integrated in the Maple Library `RAGLib` which is available at `http://www-calfor.lip6.fr/~safey`.

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