

# Using Fast Gröbner *as a preprocessing tool for global optimization problems. Applications.*

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SPACES Project  
**SCAN 2002**

# The problem

A **global optimization** problem is of the form:

(1)

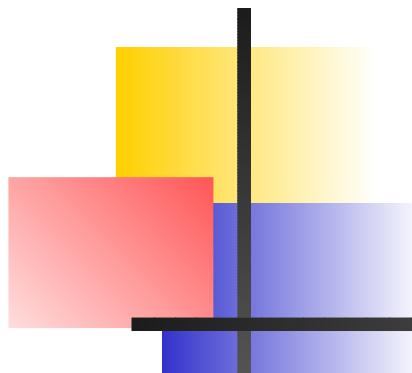
minimize  $\Phi(x)$

subject to  $f_i(x) = 0, \quad i = 1, \dots, m$

where  $\Phi : \mathbf{C}^n \rightarrow \mathbf{R}$

$f_i : \mathbf{C}^n \rightarrow \mathbf{C}$

Want **true global** minimum NOT **local** minimum.



# The problem

A **global optimization** problem is of the form:

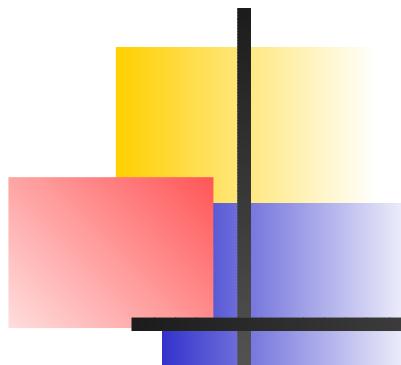
(1)

$$\text{minimize } \Phi(x)$$

Set of constraints

$$V_{\text{const}} = \{x \in \mathbf{C}^n \text{ s.t. } f_i(x) = 0 \text{ } i = 1, \dots, m\}$$

(not necessarily a bounded set).



# The problem

A **global optimization** problem is of the form:

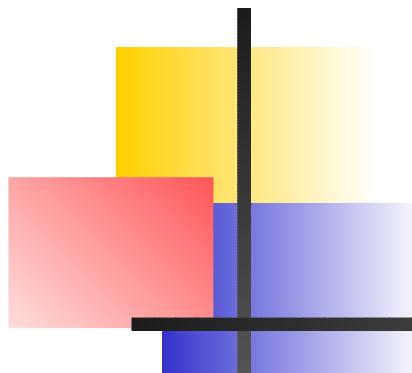
(1)

$$\text{minimize } \Phi(x)$$

Computations must be **certified**

*Hypothesis:* all the  $f_i$  are algebraic (possibly after some transformation)

Sometimes  $\Phi$  is **also** a polynomial equation.



# The problem

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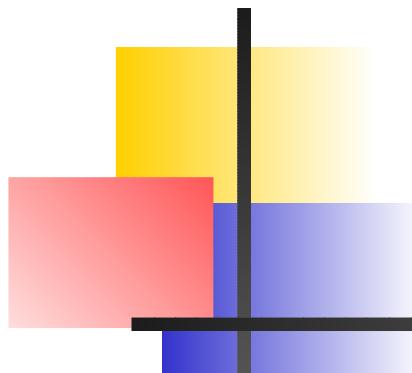
A **global optimization** problem is of the form:

(1)

$$\text{minimize } \Phi(x)$$

Two cases:

- $\Phi$  algebraic
- $\Phi$  **not** algebraic



# Solving the problem ( $\Phi$ algebraic)

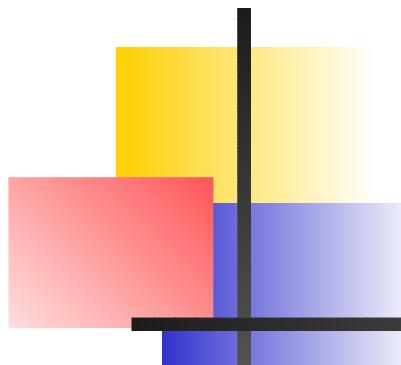
$\Phi$  and  $f_i$  are polynomials in  $\mathbf{Q}[X_1, \dots, X_n]$ .

We introduce a **new variable  $F$** .

The strategy is then to compute:

$$V = V_{\text{const}} \cup \left\{ x \text{ s.t. } \frac{\partial \Phi}{\partial x_i} = 0 \quad i = 1, \dots, n \right\} \cup \{ \Phi - F \}$$

$V$  is an **algebraic variety**.

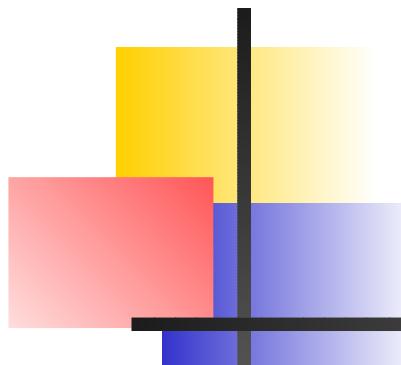


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The projection  $\pi(V) = V \cap \mathbf{Q}[F]$  is also an algebraic variety.



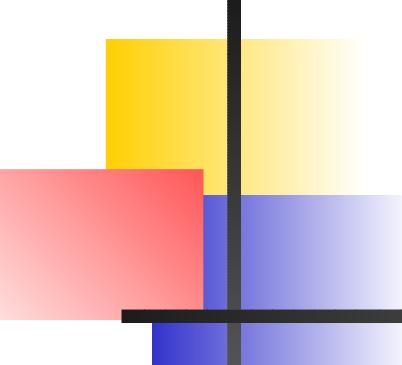
# Solving the problem ( $\Phi$ algebraic)

$\Phi$  and  $f_i$  are polynomials in  $\mathbf{Q}[X_1, \dots, X_n]$ .  
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$$V = V_{\text{const}} \cup \left\{ x \text{ s.t. } \frac{\partial \Phi}{\partial x_i} = 0 \quad i = 1, \dots, n \right\} \cup \{\Phi - F\}$$

Result is:

$$\begin{cases} \text{inconsistent} & \text{if } V = \emptyset \\ \text{finite} & \text{if } \pi(V) = \{f(F)\} \\ \text{infinite} & \text{if } V \neq \emptyset \text{ and } \pi(V) = \emptyset \end{cases}$$



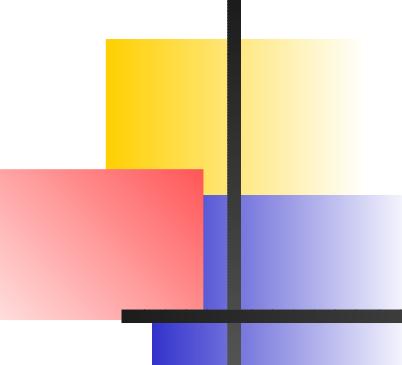
# What can we compute ?

Computer Algebra = Exact Computation

Tool to study algebraic variety: **Gröbner bases**  
**(Buchberger)**

To compute Gröbner bases:

- original Buchberger algorithm (Maple, Mathematica, ...)
- More efficient algorithms (F4, F5 **Faugère**):  
> 1000 speedup

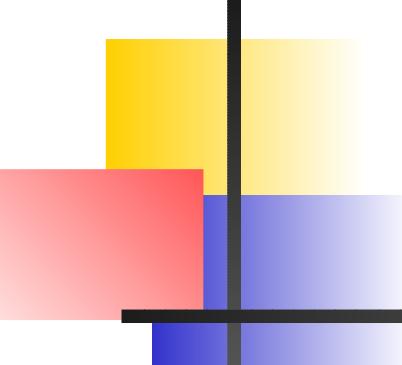


# What can we compute ?

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algebraic variety  $V$

- Decide if  $V =$  (complex roots)
- Decide if  $V =$  (real roots) **Rouillier, ...**
- Count the number of solutions. (degree  $V$ )
- Find all the solutions if **finite**.
- Count the number of free parameters if **infinite**. ( $\dim V$ )
- Find equations of curves, surfaces if **infinite**.



# What can we compute ?

System of polynomial equations  $f_1 = 0, \dots, f_m = 0$   
 $f_i \in \mathbf{Q}[x_1, \dots, x_n]$ .

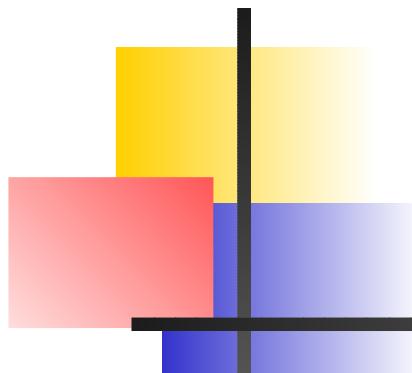
We associate  $\mathcal{I}(F) = Id(f_1, \dots, f_m)$

**Gröbner basis:** an equivalent basis of the ideal  $\mathcal{I}(F)$

If  $n = m$  the **shape** of (lexico) Gröbner:

$$\left\{ \begin{array}{l} Q_n(x_n) = 0 \\ x_{n-1} = Q_{n-1}(x_n) \\ \dots \\ x_1 = Q_1(x_n) \end{array} \right.$$

where all the  $Q_i$  are univariate polynomials.



# What can we compute ?

Algorithm  $F7 \longrightarrow$  possible to compute (**efficiently**)  
a decomposition into **irreducible** varieties

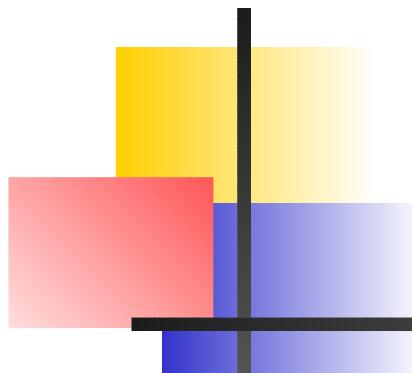
$$V = V_1 \cup \dots \cup V_k$$

each  $V_i$  is an **irreducible** variety.

*“generalization of factorization”*

Now for each component  $V_i$  we compute the  
minimum of  $\pi(V_i)$ .

We can separate the components: points, curves, .

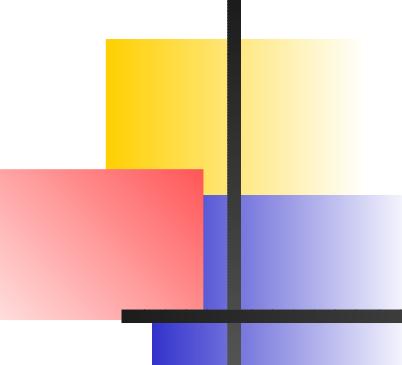


# Small example Step by step

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Minimize ( $x, y$  and  $z$  reals):

$$\Phi = x^4 + 2y^4 + 7z^4 - 23xyz + x + y + z$$

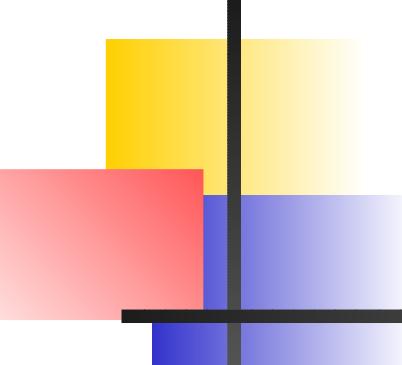


# Small example Step by step

$$\Phi = x^4 + 2y^4 + 7z^4 - 23xyz + x + y + z$$

We have to study:

$$V = \left\{ \frac{\partial \Phi}{\partial x}, \frac{\partial \Phi}{\partial y}, \frac{\partial \Phi}{\partial z}, \Phi - \textcolor{red}{F} \right\}$$



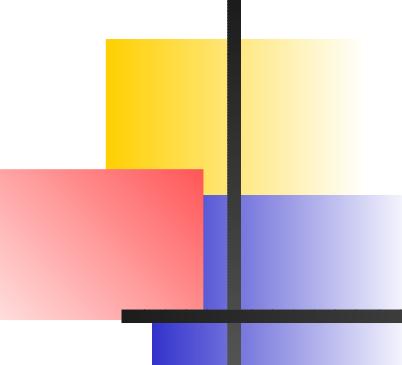
# Small example Step by step

$$\Phi = x^4 + 2y^4 + 7z^4 - 23xyz + x + y + z$$

that is to say

$$V = \left\{ \begin{array}{l} 4x^3 - 23yz + 1, \\ 8y^3 - 23xz + 1, \\ 28z^3 - 23xy + 1, \\ \textcolor{red}{F} - x^4 - 2y^4 - 7z^4 + 23xyz - x - y - z \end{array} \right\}$$

We compute  $\pi(V)$  with Gröbner:



# Small example Step by step

$$\Phi = x^4 + 2y^4 + 7z^4 - 23xyz + x + y + z$$

174074817067741462271717846167661155823151248359104319816554512384  $\textcolor{red}{F^{27}}$  +

217469959299347493498124079416984221012082448625268356972251925446656  $F^{26}$  +

127351445759532718878366086442790767422381761661015043840970879748014080  $F^{25}$  +

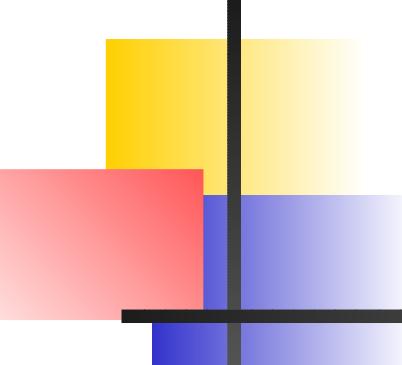
...

-7052816420206894754768280952881688901743611756955198695128318907825763244202353845465337337

55729214128592877315801583324424596712260412207699253643958808719706505824920812659947564061

45026365567011686407562456211323862755134104157993558883560334542365052764323056683885563268

5534792477208943881900750569211859763481899524701899662119138642031450645851004004524463784

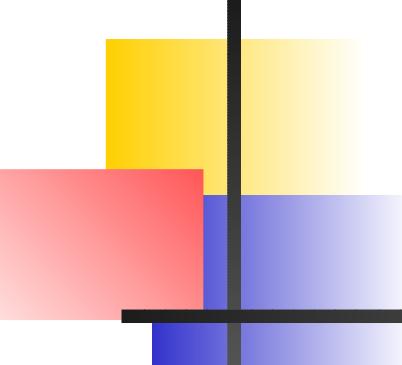


# Small example Step by step

$$\Phi = x^4 + 2y^4 + 7z^4 - 23xyz + x + y + z$$

Real solutions (isolation of real roots: intervals)  
Uspensky Algorithm ([Rouillier, Zimmermann](#))

$$[[[-\frac{1404247785931}{17179869184}, -\frac{2808495571861}{34359738368}], [-\frac{2762198617459}{34359738368}, -\frac{1381099308729}{17179869184}], [-\frac{1364825393905}{17179869184}, -\frac{2729650787809}{34359738368}], [-\frac{2433009117369}{34359738368}, -\frac{304126139671}{4294967296}], [-\frac{9322721949}{17179869184}, -\frac{18645443897}{34359738368}], [-\frac{15956333217}{34359738368}, -\frac{498635413}{1073741824}], [\frac{7493386665}{17179869184}, \frac{14986773331}{34359738368}]]$$



# Small example Step by step

$$\Phi = x^4 + 2y^4 + 7z^4 - 23xyz + x + y + z$$

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Uspensky Algorithm ([Rouillier, Zimmermann](#))

[ **-81.73797896** , -80.39056025, -79.44329373,  
-70.80988485, -0.5426538380, -0.4643904167,  
0.4361725101]

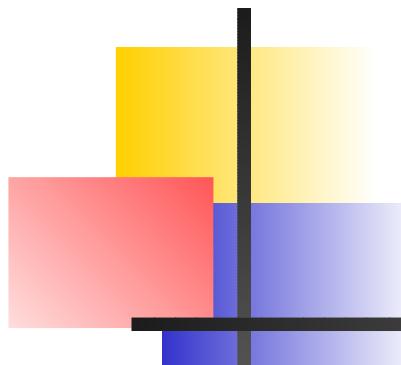
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Hence, is the **global minimum.**



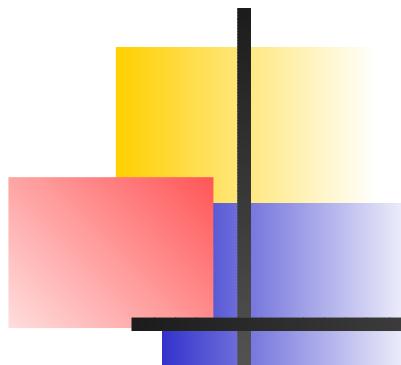
# Solving the problem ( $\Phi$ not algebraic)

→ compute irreducible varieties of

$$V_{\text{const}} = V_1 \cup \dots \cup V_k$$

For  $i = 1, \dots, k$ ,

$V_i$  is an irreducible variety of dimension  $d_i$ .



# Solving the problem ( $\Phi$ not algebraic)

$$V_{\text{const}} = V_1 \cup \dots V_k$$

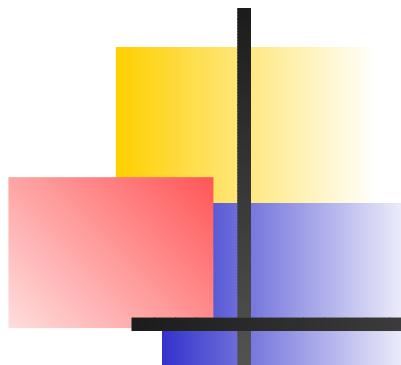
$V_i$  is an irreducible variety of dimension  $d_i$ .

Usually  $V_i = \{H_i(x_1, \dots, x_{d_i}, \mathbf{x_n})\}$

$x_{d_i}, \dots, x_1$  are free parameters

it is possible to find

$$\Phi_i(x_1, \dots, x_{d_i}, \mathbf{x_n}) = \Phi \text{ on } V_i$$

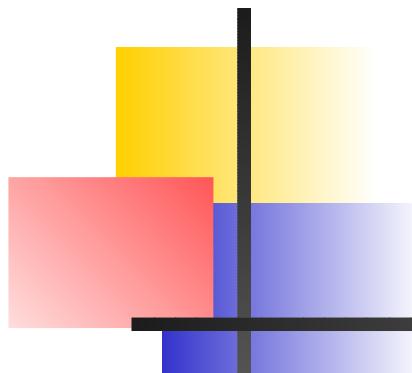


# Solving the problem ( $\Phi$ not algebraic)

$$V_{\text{const}} = V_1 \cup \dots \cup V_k$$

$V_i$  is an irreducible variety of dimension  $d_i$ .  
We now use a **numerical** (global) optimization program to minimize  $\Phi_i$  on the curve (surface, ...)  $H_i$ .

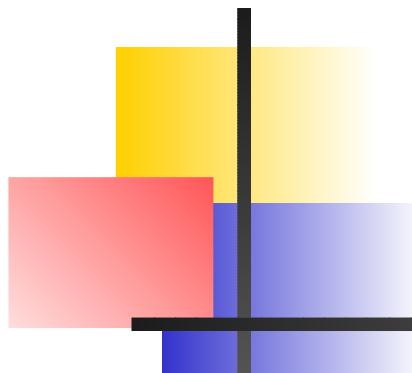
we have thus reduce the number of unknowns from  $n$  to  $d_i$ .



# Applications

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- the test problem of R. Baker Kearfott
- Design of filter bank (with Rouillier, Moreau)
- Breguet Formula



# The problem of R. Baker Kearfott

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Paper of [R. Baker Kearfott](#) (Journal of Reliable Computing 2001) we found:

# The problem of R. Baker Kearfott

Find  $a_1, a_2, a_3, x_1, x_2$  and  $x_3$  such that  $c_i = 0$  for  $i = 1, \dots, 6$ .

$$c_1 = 0.08413a_1 + 0.08413a_2 + 0.08413a_3 + 0.08413 + 0.2163q_1 + 0.0792q_2 - 0.1372q_3$$

$$c_2 = -0.3266a_1 - 0.3266a_2 - 0.3266a_3 - 0.3266 - 0.57q_1 - 0.0792q_2 + 0.4907q_3$$

$$c_3 = 0.2704a_1 + 0.2704a_2 + 0.2704a_3 + 0.2704 + 0.3536a_1(x_1 - x_3) + 0.3536a_2(x_1^2 - x_3^2) + 0.3536a_3(x_1^3 - x_3^3) + 0.3536x_1^4 - 0.3536x_3^4$$

$$c_4 = 0.02383p_1 - 0.01592a_1 - 0.01592a_2 - 0.01592a_3 - 0.01592 - 0.08295q_1 - 0.05158q_2 + 0.0314q_3$$

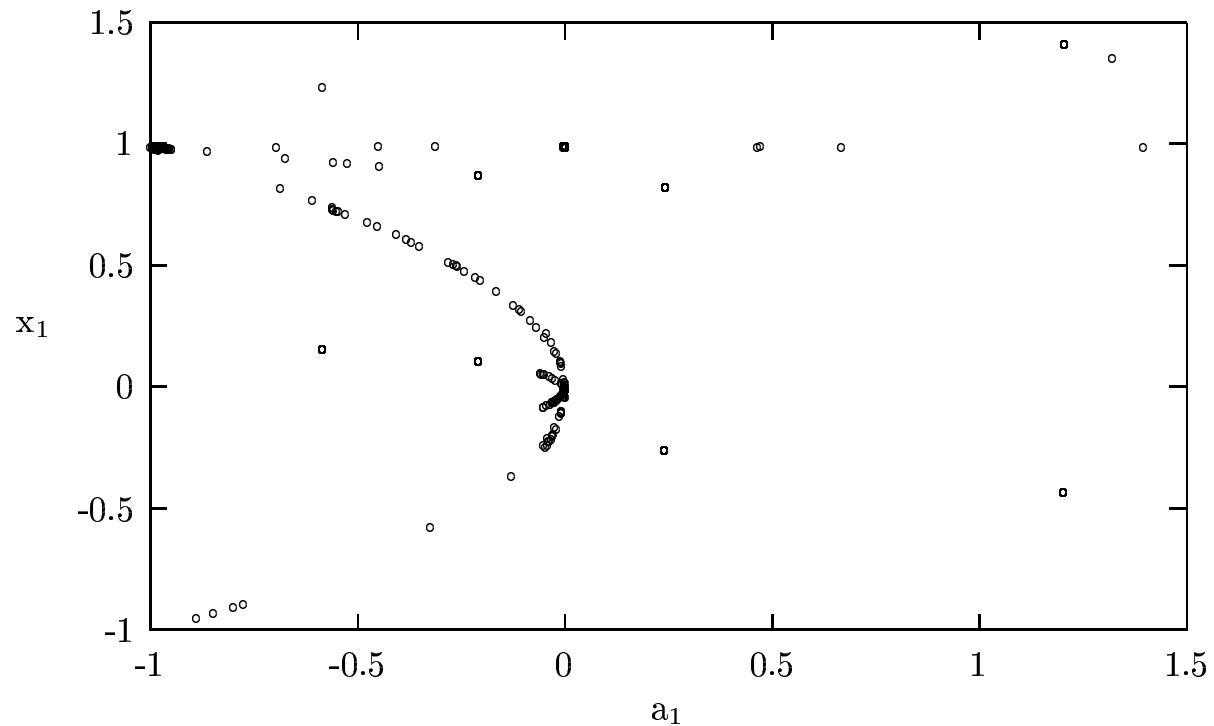
$$c_5 = -0.04768p_2 - 0.06774a_1 - 0.06774a_2 - 0.06774a_3 - 0.06774 - 0.1509q_1 + 0.1509q_3$$

$$c_6 = 0.02383p_3 - 0.1191a_1 - 0.1191a_2 - 0.1191a_3 - 0.1191 - 0.0314q_1 + 0.05158q_2 + 0.08295q_3$$

where

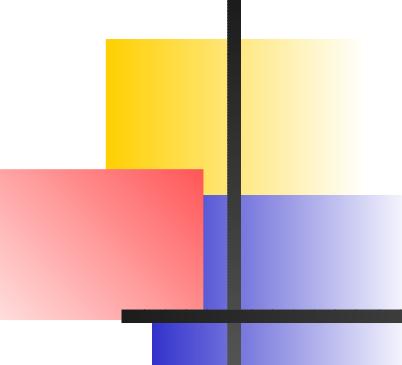
$$i = 1, 2, 3 \left\{ \begin{array}{l} q_i := a_1x_i + a_2x_i^2 + a_3x_i^3 + x_i^4 \\ p_i := \frac{\partial q_i}{\partial x_i} = a_1 + 2a_2x_i + 3a_3x_i^2 + 4x_i^3 \\ r = q_i(1) = a_1 + a_2 + a_3 + 1 \end{array} \right.$$

# The problem of R. Baker Kearfott



Solution found by **GlobSol**

*“A run of one hour produced 216 unverified small boxes and 12 boxes with verified feasible points”.*



# Decomposition into irreducible varieties

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We used the  $F_7$  algorithm to compute the decomposition: it takes less than 1 minute on a PC (*Pentium III - 600 Mhz*).

$$V = V_1 \cap V_2 \cap V_3 \cap V_4 \cap \cdots \cap V_{12}$$

$\dim(V_i) = 1$  for  $i = 1, 2, 3$  and  $\dim(V_i) = 0$  for  $i = 4, \dots, 12$ .

There are 26 real isolated points and 64 (pure) complex isolated points.

# Decomposition into irreducible varieties

First two components parametrized by  $a_1$  or  $a_2$ :

$$V_1 = [x_3 = x_2 = x_1 = 1, a_2 = -2\textcolor{red}{a}_1 + 1, a_3 = \textcolor{red}{a}_1 - 2]$$

$$V_2 = [a_3 = -\textcolor{red}{a}_2 - 1, x_3 = x_2 = x_1 = a_1 = 0]$$

$$\begin{aligned} V_3 = & [a_2 - 2x_1 + \textcolor{red}{a}_1, a_3 + 2x_1 + 1, x_3 - x_1, \\ & x_2 - x_1, x_1^2 + \textcolor{red}{a}_1] \end{aligned}$$

# Decomposition into irreducible varieties

All the other components are zero dimensional  
**(isolated points).**

$$V_4 = r_1 = [a_2 = -a_3 = 3, x_3 = x_2 = x_1 = -a_1 = 1]$$

$$V_5 = r_2 = [a_3 = -1, a_2 = a_1 = x_3 = x_2 = x_1 = 0]$$

$$V_6 = r_3 = [a_3 = -2, a_2 = x_3 = x_2 = 1, x_1 = a_1 = 0]$$

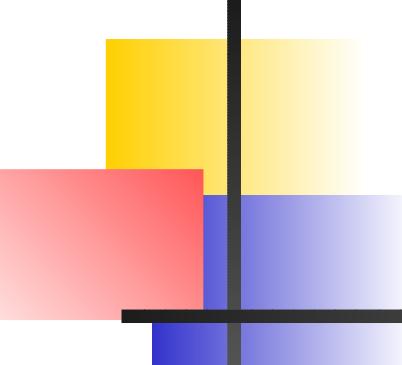
$$V_7 = r_4 = [a_3 = -2, a_2 = x_1 = 1, x_3 = x_2 = a_1 = 0]$$

$$V_8 = r_5 = [a_3 = -2, a_2 = x_3 = 1, x_2 = x_1 = a_1 = 0]$$

$$V_9 = r_6 = [a_3 = -2, a_2 = x_3 = x_1 = 1, x_2 = a_1 = 0]$$

$$V_{10} = r_7 = [a_3 = -2, a_2 = x_2 = x_1 = 0, x_3 = a_1 = 0]$$

$$V_{11} = r_8 = [a_3 = -2, a_2 = x_2 = 1, x_3 = x_1 = a_1 = 0]$$



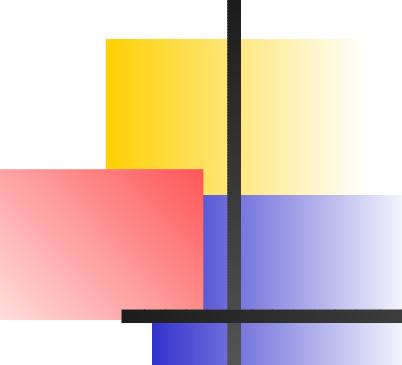
# Decomposition into irreducible varieties

The shape of  $V_{12}$  is

$$V_{12} = \left\{ \begin{array}{l} x_2 = Q_1(x_3) \\ x_1 = Q_2(x_3) \\ a_1 = Q_3(x_3) \\ a_2 = Q_4(x_3) \\ a_3 = Q_5(x_3) \\ Q(x_3) = 0 \end{array} \right.$$

where  $Q$  (resp.  $Q_i$ ) is a univariate polynomial of degree 72 (resp. 71).

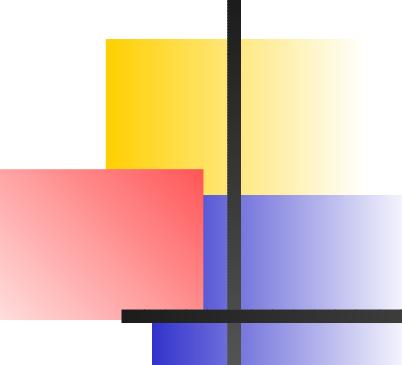
# Decomposition into irreducible varieties



# Decomposition into irreducible varieties

There are **exactly 18 real roots.**

$$\begin{aligned} & [[-12255552113270/2^{33}, -12255552113269/2^{33}], \\ & [-17317363802/2^{33}, -17317363801/2^{33}], [-7816254912/2^{33}, -7816254911/2^{33}], \\ & [-3657648718/2^{33}, -3657648717/2^{33}], [-2145107364/2^{33}, -2145107363/2^{33}], \\ & [1009266551/2^{33}, 1009266552/2^{33}], [1424585688/2^{33}, 1424585689/2^{33}], \\ & [3438423730/2^{33}, 3438423731/2^{33}], [5148036309/2^{33}, 5148036310/2^{33}], \\ & [7163775870/2^{33}, 7163775871/2^{33}], [7582095283/2^{33}, 7582095284/2^{33}], \\ & [10716772156/2^{33}, 10716772157/2^{33}], [12218728088/2^{33}, 12218728089/2^{33}], \\ & [16171083098/2^{33}, 16171083099/2^{33}], [25944225474/2^{33}, 25944225475/2^{33}], \\ & [946227867734/2^{33}, 946227867735/2^{33}], [7270849565383/2^{33}, 7270849565384/2^{33}], \\ & [21249066182943/2^{33}, 21249066182944/2^{33}]] \end{aligned}$$



# Decomposition into irreducible varieties

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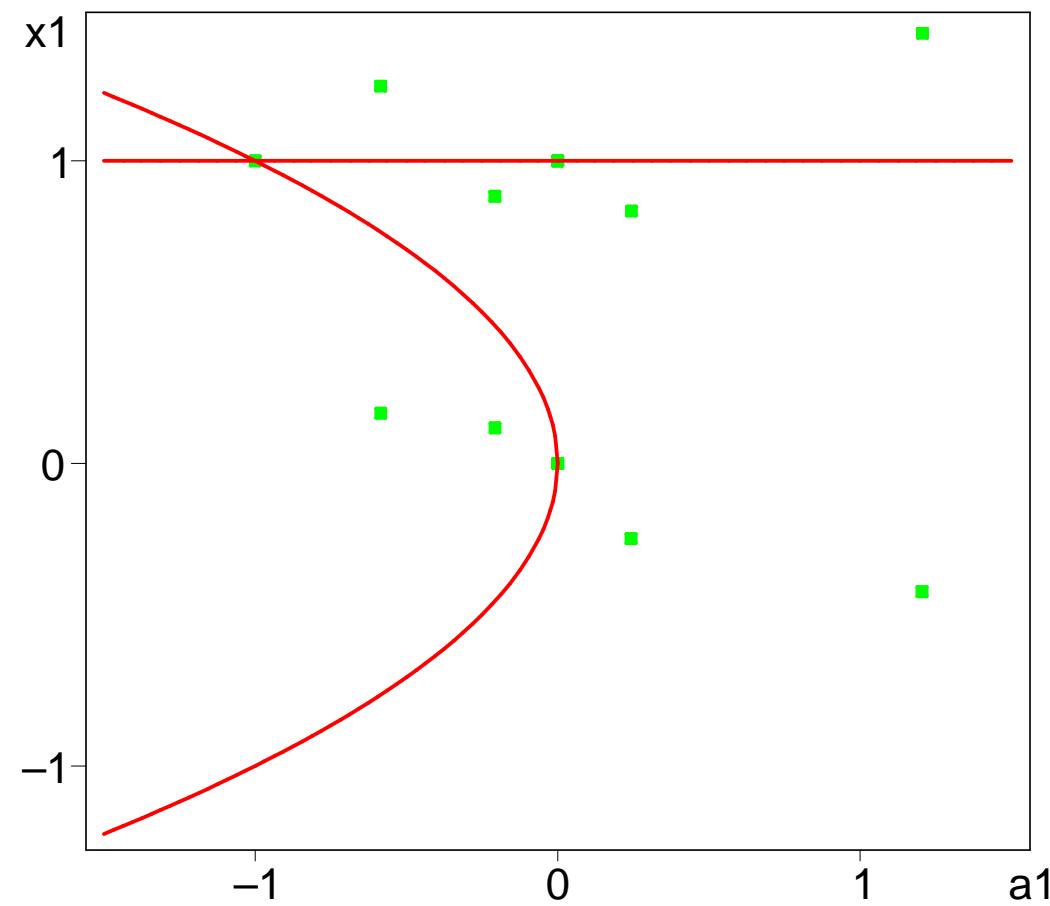
$$r_9 = [a_1 = -.616087e9, a_2 = 370507., a_3 = 2018.77, x_2 = -135.308, x_1 = 178.797, x_3 = -]$$

...

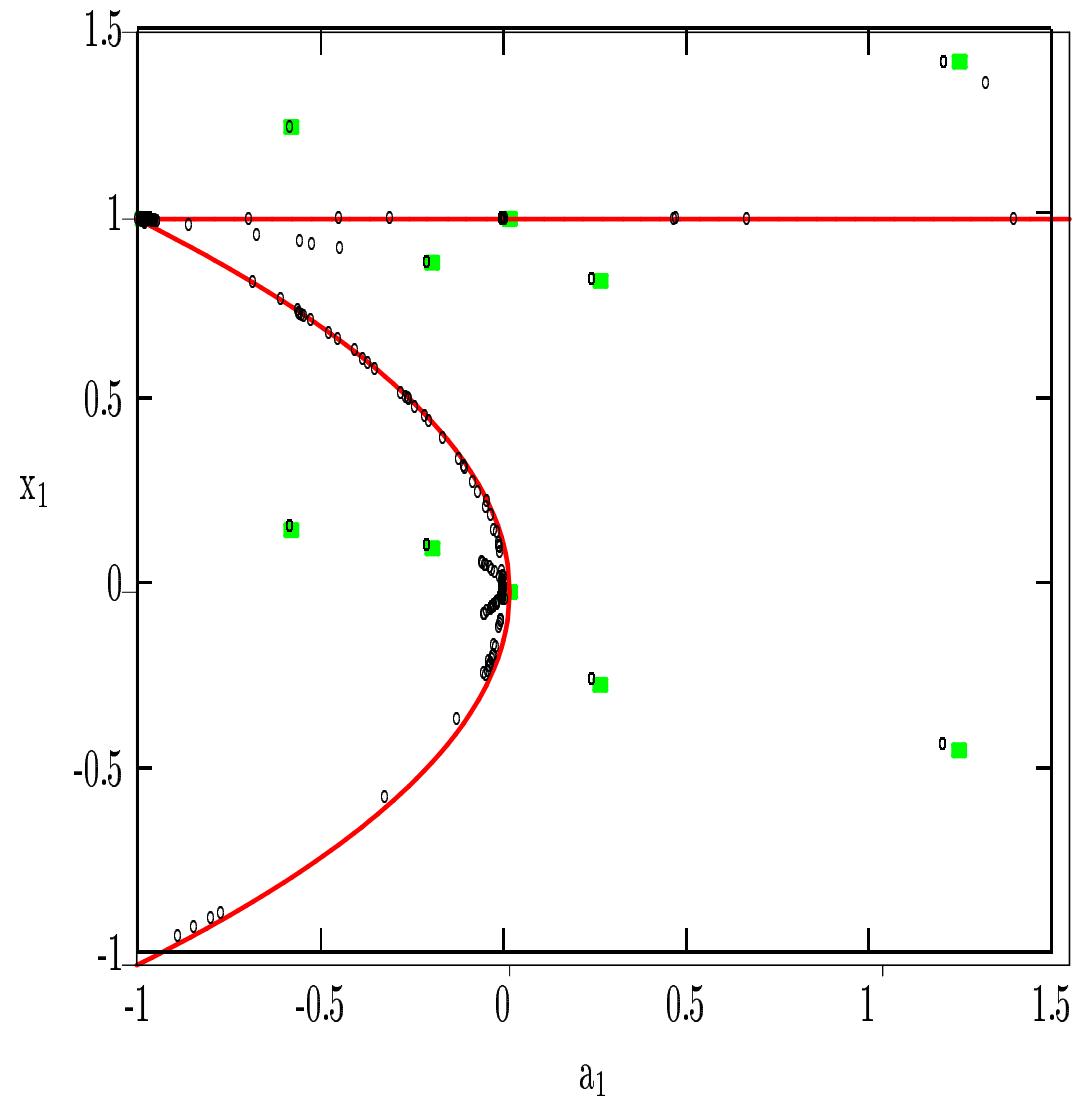
$$r_{26} = [a_2 = .411203e8, a_1 = -.255363e11, x_1 = 197.131, x_2 = 2255.12, a_3 = -15159.3, x_3 = ]$$

Hence we have found explicitly  $(r_i)_{i=1,\dots,26}$  all the real isolated roots of the system.

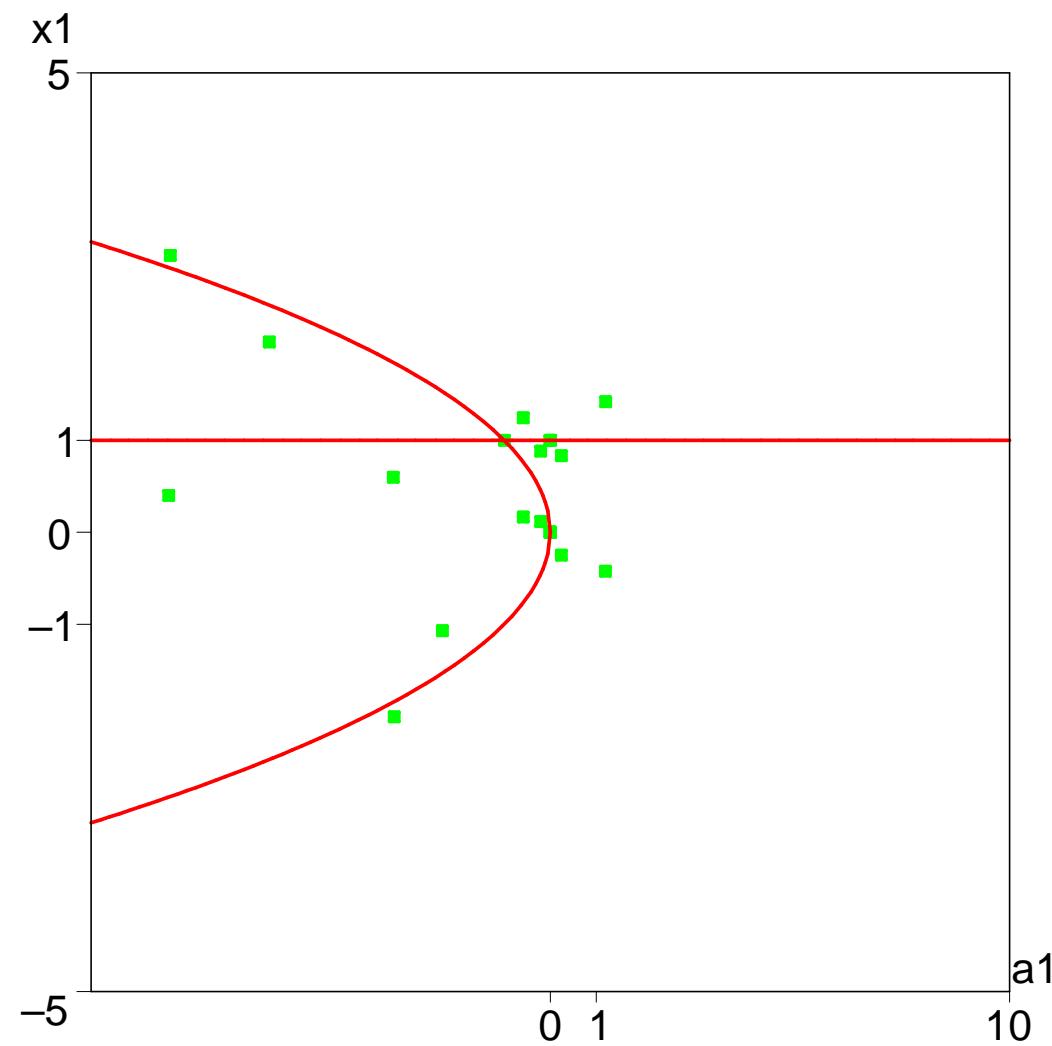
# Solutions in $[-\frac{3}{2}, \frac{3}{2}]$



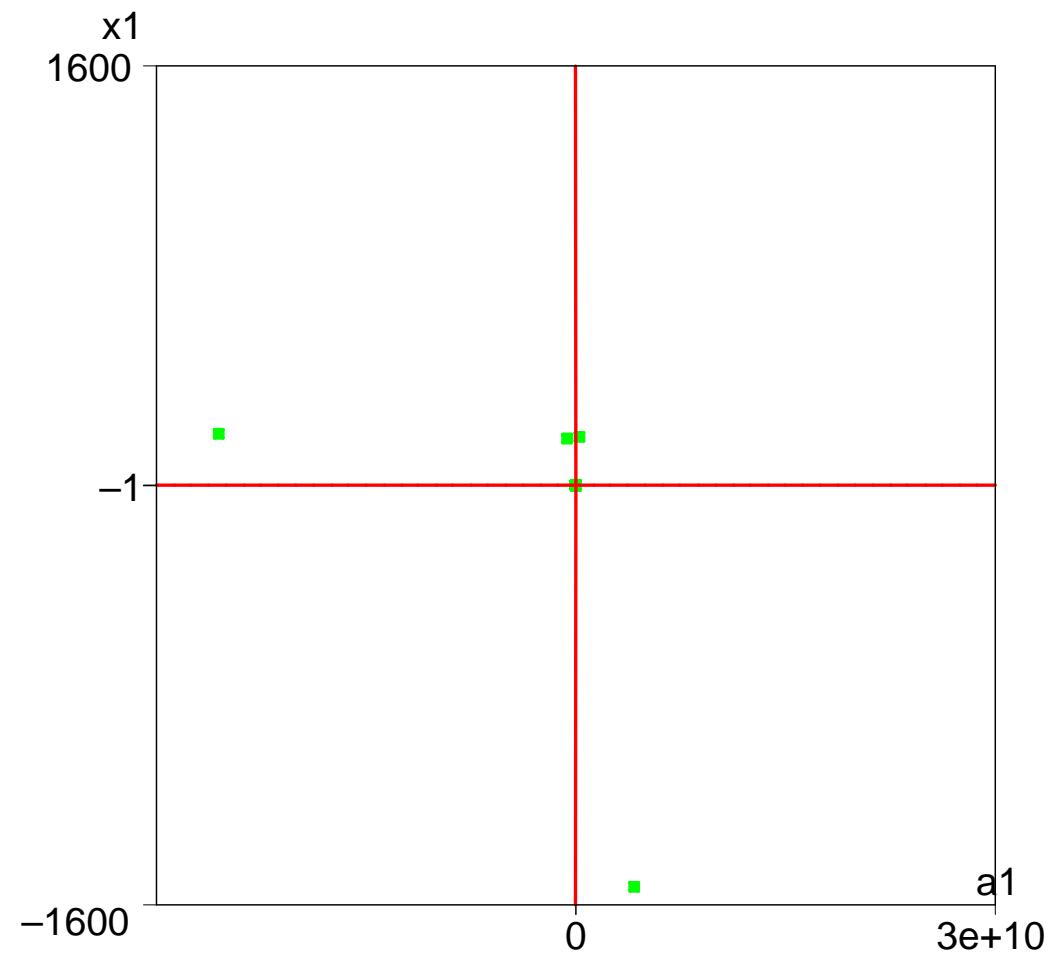
# Comparison

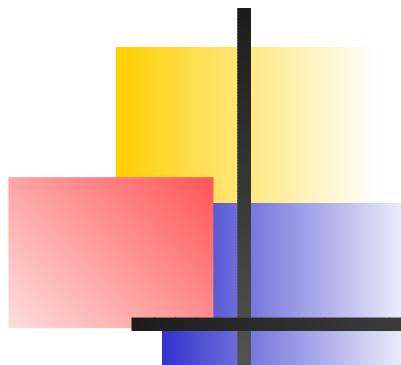


# Solutions in [-10,10]



# All real Solutions





# Design of filter bank

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(with F. Moreau and F. Rouillier)

This problem was supplied to us by **French Telecom**: design of **two-dimensional filter banks** yielding orthogonality and linear-phase filters, and generating **regular wavelet bases**.

Using cascade (Vetterli and Kovacevic) forms implies dealing with *global non-linear optimization*.

# Design of filter bank

$K$  be a given integer (in practice  $k = 7, 8, 9$ ).  
Let  $(\alpha_i)_{i=1,\dots,K}$  and  $\beta_i$  are **free** parameters (real numbers).

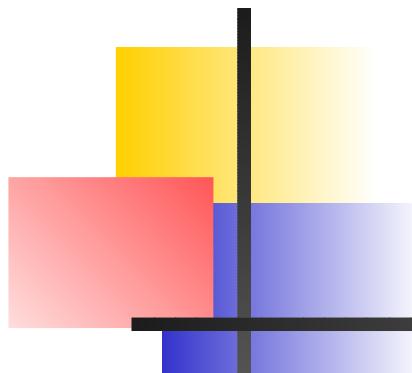
$$z = (z_1, z_2):$$

# Design of filter bank

$$R_i = \begin{bmatrix} \cos \alpha_i & -\sin \alpha_i & 0 & 0 \\ \sin \alpha_i & \cos \alpha_i & 0 & 0 \\ 0 & 0 & \cos \beta_i & -\sin \beta_i \\ 0 & 0 & \sin \beta_i & \cos \beta_i \end{bmatrix} \quad W = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad D(z_1, z_2) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & z_1 & 0 & 0 \\ 0 & 0 & z_2 & 0 \\ 0 & 0 & 0 & z_1 z_2 \end{bmatrix}$$

$$\mathcal{H}(z) = R_1 W P \prod_{i=2}^K (D(z_1, z_2) P W R_i W P)$$

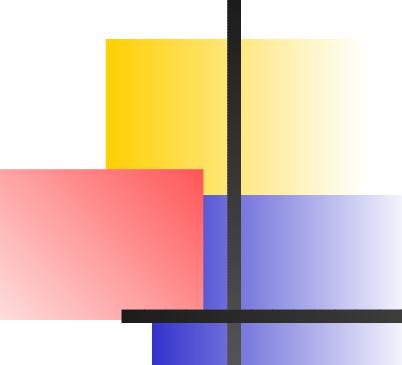


# Design of filter bank

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$$\left( \mathcal{H}_{i,j} \right)_{i,j} = R_1 WP \prod_{i=2}^K \left( D(z_1, z_2) PWR_i WP \right)$$

$$H_i(z) = \mathcal{H}_{i,0}(z_1^2, z_2^2) + \mathcal{H}_{i,1}(z_1^2, z_2^2)z_1 + \mathcal{H}_{i,2}(z_1^2, z_2^2)z_2 + \mathcal{H}_{i,3}(z_1^2, z_2^2)z_1z_2$$



# Design of filter bank

$$H_i(z) = \mathcal{H}_{i,0}(z_1^2, z_2^2) + \mathcal{H}_{i,1}(z_1^2, z_2^2)z_1 + \mathcal{H}_{i,2}(z_1^2, z_2^2)z_2 + \mathcal{H}_{i,3}(z_1^2, z_2^2)z_1z_2$$

The **goal** is to find the maximum integer  $N$  such that for all  $k_1, k_2, k_1 + k_2 \leq N$

$$\frac{\partial^{k_1+k_2} H_0}{\partial z_1^{k_1} \partial z_2^{k_2}} = 0 \text{ at } (1, -1), (-1, -1), (-1, 1)$$
$$\frac{\partial^{k_1+k_2} H_2}{\partial z_1^{k_1} \partial z_2^{k_2}} = 0 \text{ at } (1, 1), (-1, -1), (-1, 1)$$

$$\frac{\partial^{k_1+k_2} H_1}{\partial z_1^{k_1} \partial z_2^{k_2}} = 0 \text{ at } (1, -1), (1, 1), (-1, 1)$$
$$\frac{\partial^{k_1+k_2} H_3}{\partial z_1^{k_1} \partial z_2^{k_2}} = 0 \text{ at } (1, -1), (-1, -1), (1, 1)$$

MIN

$\min \left\{ \text{Energy Compaction}(\alpha_i, \beta_i) \right\}$

# Design of filter bank

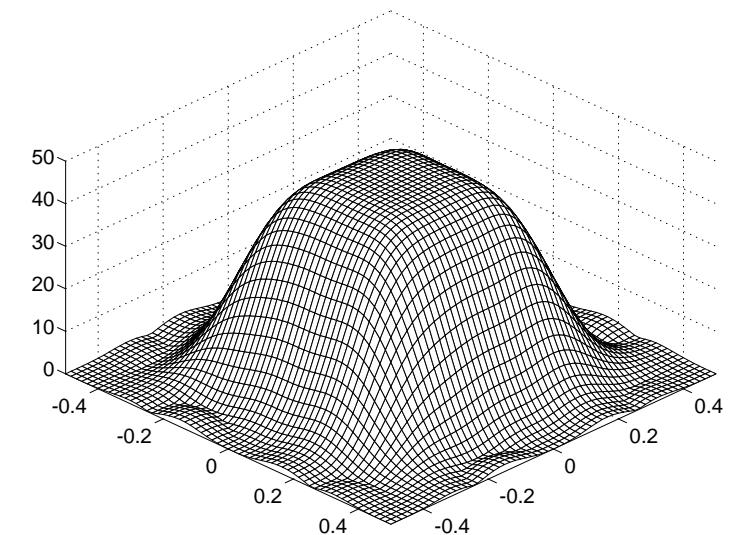
It is obvious that we can transform all the equations except (MIN) into **polynomials** by adding new variables  $c_i = \cos(\alpha_i)$ ,  $s_i = \sin(\alpha_i)$  and a new equation  $c_i^2 + s_i^2 = 1$  (the same for the  $\beta_i$ ).

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The first task is to find the maximum  $N$  for a given  $K$ . This is done with **gröbner bases** methods: for a given  $K$  we try successively  $N = 1, 2, 3$  until we found no solution.

# Design of filter bank

$K$	3	4	5	6	7	8
maximal $N$	2	2	3	3	4	5
dimension	0	2	2	4	2	2



# Design of filter bank

$K$	3	4	5	6	7	8
maximal $N$	2	2	3	3	4	5
dimension	0	2	2	4	2	2

we have a two dimensional system.

We use now a “standard” optimizer to optimize the criteria (MIN) but with respect to **only two parameters**.

# Breguet Formula

The goal (problem submitted by ONERA) is to  
**maximize** the Breguet endurance formula

$$\psi = \psi_1 \psi_2$$

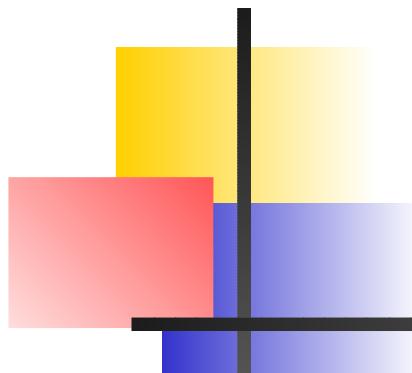
$$\psi_1 = \frac{1}{\sqrt{\frac{K_r (k_f + k_e + k_t + k_{cu}) \sqrt{S} L^{3/2} + \frac{\sqrt{2} \mu_{mot} g^{3/2} C_x_0 \sqrt{S}}{v_{prop} \sqrt{\rho} C_z^{3/2} \sqrt{m_{init}}} + \frac{\sqrt{2} \mu_{mot} g^{3/2} K_r C_x_0 S L^{3/2}}{v_{prop} \sqrt{\rho} C_z^{3/2} \sqrt{m_{init}}}}{+\frac{m_s S}{m_{init}} + \frac{\sqrt{2} \mu_{mot} g^{3/2} \sqrt{C_z} K_r S \sqrt{L}}{v_{prop} \sqrt{\rho} \sqrt{m_{init}} \pi} + \frac{\sqrt{2} \mu_{mot} g^{3/2} \sqrt{C_z} \sqrt{S}}{v_{prop} \sqrt{\rho} \sqrt{m_{init}} \pi L}} + k_f + k_e + k_t + k_{cu}}} - 1$$

# Breguet Formula

$$\psi = \psi_1 \psi_2$$

$$\psi_1 = \frac{1}{\sqrt{\frac{K_r (k_f + k_e + k_t + k_{cu}) \sqrt{S} L^{3/2} + \frac{\sqrt{2} \mu_{mot} g^{3/2} Cx_0 \sqrt{S}}{v_{prop} \sqrt{\rho} C_z^{3/2} \sqrt{m_{init}}} + \frac{\sqrt{2} \mu_{mot} g^{3/2} K_r Cx_0 S L^{3/2}}{v_{prop} \sqrt{\rho} C_z^{3/2} \sqrt{m_{init}}}}{+\frac{m_s S}{m_{init}} + \frac{\sqrt{2} \mu_{mot} g^{3/2} \sqrt{C_z} K_r S \sqrt{L}}{v_{prop} \sqrt{\rho} \sqrt{m_{init}} \pi} + \frac{\sqrt{2} \mu_{mot} g^{3/2} \sqrt{C_z} \sqrt{S}}{v_{prop} \sqrt{\rho} \sqrt{m_{init}} \pi L} + k_f + k_e + k_t + k_{cu}}}} - 1$$

$$\psi_2 = \sqrt{S} \left( \frac{1/2 \frac{C_{x0} C_{sp} g^{3/2} \sqrt{m_{init}} \sqrt{2}}{v_{prop} \sqrt{\rho} C_z^{3/2}} + 1/2 \frac{\sqrt{C_z} C_{sp} g^{3/2} \sqrt{m_{init}} \sqrt{2}}{\pi v_{prop} \sqrt{\rho} L}} \right)^{-1}$$



# Breguet Formula

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$$\psi = \psi_1 \psi_2$$

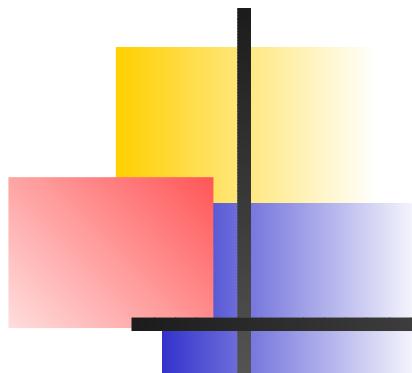
- transform  $\psi$  rational fraction
- in a few seconds **96 complex** solutions (local extrema)
- plugging these values into  $\psi$  to find the global maximum.

# Breguet Formula

$$\begin{array}{l} \psi = \psi_1 \psi_2 \\ \\ L = 44.42\dots \\ S = 132.87\dots \\ L = 44.53\dots \\ S = 131.73\dots \end{array} \left. \begin{array}{c} \\ \\ \\ \end{array} \right\} \rightarrow \begin{array}{l} \psi_0 = 245.962534835\dots \\ \\ \psi = \psi_0 + \varepsilon \text{ with } 3.61\dots \cdot 10^{-} \end{array}$$

solution found by numerical methods not optimal

(a linearization of the equation  $\psi$  was first computed).



# Future work

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Inexact data:

- Replace the exact arithmetic
  - double
  - Multiprecision floats (Zimmermann, GMP)
  - mpfi (Revol, Rouillier)
- Infinitesimal deformations