Ridges and umbilics of polynomial parametric surfaces

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Computational Methods for Algebraic Spline Surfaces II
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Outline

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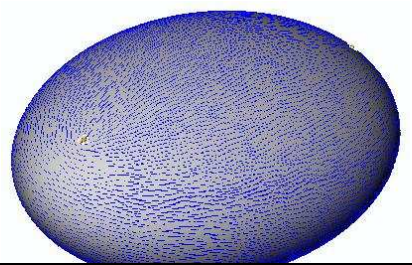
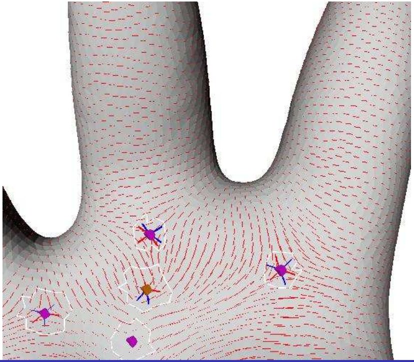
Principal curvatures and directions

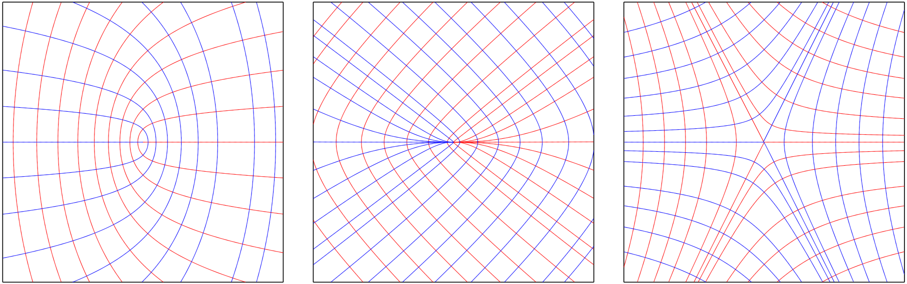
- k_1 and d_1 maximal principal curvature and direction (Blue).
- k_2 and d_2 minimal principal curvature and direction (Red).
- k_i and d_i are eigenvalues and eigenvectors of the Weingarten map $W = I^{-1}II$.
- d_1 and d_2 are orthogonal.





Umbilics and curvature lines

- A curvature line is an integral curve of the principal direction field.
- Umbilics are singularities of these fields, $k_1 = k_2$

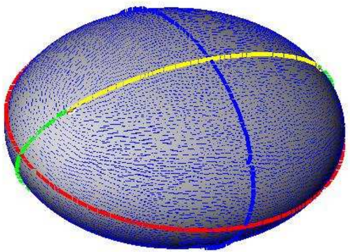


Ridges

- A blue (red) ridge point is a point where k_1 (k_2) has an extremum along its curvature line.

$$< \nabla k_1, d_1 >= 0 \quad (< \nabla k_2, d_2 >= 0) \tag{1}$$

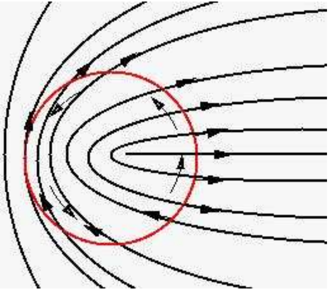
- Ridge points form lines going through umbilics.



Umbilics, ridges, and principal blue foliation on the ellipsoid

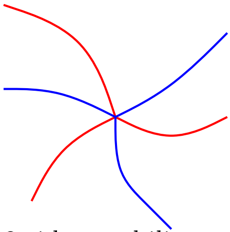
Orientation of principal directions

- Principal directions d_1 (d_2) are not globally orientable.
- The sign of $< \nabla k_1, d_1 >$ is not well defined.
- $< \nabla k_1, d_1 >= 0$ cannot be a global equation of blue ridges.

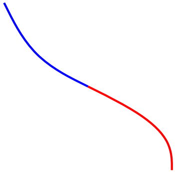


The principal field is not orientable around an umbilic

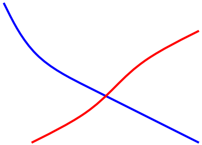
Singularities of the ridge curve



3-ridge umbilic



1-ridge umbilic

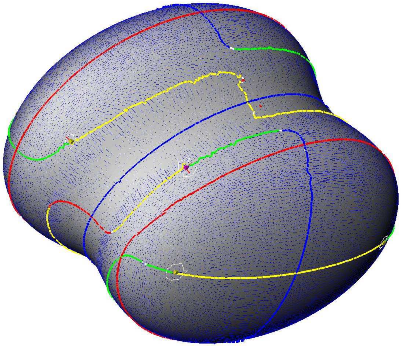


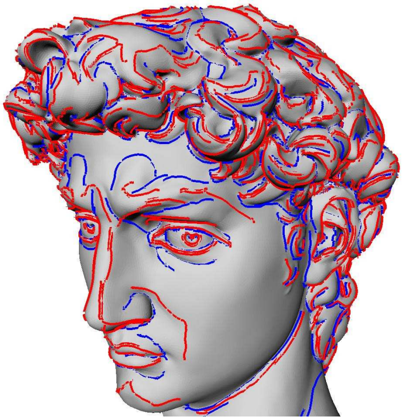
Purple point

Difficulties of ridge extraction

- Need third order derivatives of the surface.
- Singularities: Umbilics and Purple points
- Orientation problem.

Illustrations: ridges and crest lines





Outline

Problem statement

- The surface is parametrerized:

$$\Phi : (u, v) \in \mathbb{R}^2 \longrightarrow \Phi(u, v) \in \mathbb{R}^3$$

- Find a well defined function

$$P : (u, v) \in \mathbb{R}^2 \longrightarrow P(u, v) \in \mathbb{R}$$

such that $P = 0$ is the ridge curve in the parametric domain.

Solving the orientation problem

- Consider blue and red ridges together
  $< \nabla k_1, d_1 > \times < \nabla k_2, d_2 >$ is orientation independant.
- Find two vector fields v_1 and w_1 orienting d_1 such that:
  $v_1 = w_1 = 0$ characterizes umbilics.
  Note: each vector field must vanish on some curve joining umbilics
- v_1 and w_1 are computed from the two dependant equations of the eigenvector system for d_1.

Some technicallities

- $p_2 = (k_1 - k_2)^2 = 0$ characterize umbilics.
  It is a smooth function of the second derivatives of Φ.
- Define a, a', b, b' such that: $\langle \text{Numer}(\nabla k_1), v_1 \rangle = a\sqrt{p_2} + b$ and $\langle \text{Numer}(\nabla k_1), w_1 \rangle = a'\sqrt{p_2} + b'$.
  These are smooth function of the derivatives of Φ up to the third order.

Main result

The ridge curve has equation $P = ab' - a'b = 0$.
For a point of this set one has:

- If $p_2 = 0$, the point is an umbilic.
- If $p_2 \neq 0$ then
  - If $ab \neq 0$ or $a'b' \neq 0$ then the sign of one these non-vanishing products gives the color of the ridge point.
  - Otherwise, $a = b = a' = b' = 0$ and the point is a purple point.

Singularities of the ridge curve

- 1-ridge umbilics

$$S_{1R} = \{p_2 = P = P_u = P_v = 0, \delta(P_3) < 0\}$$

- 3-ridge umbilics

$$S_{3R} = \{p_2 = P = P_u = P_v = 0, \delta(P_3) > 0\}$$

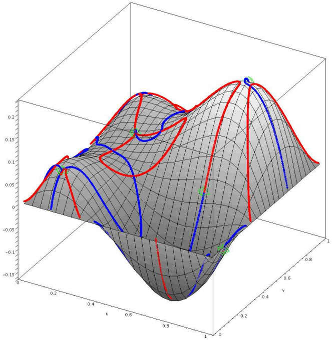
- Purple points

$$S_p = \{a = b = a' = b' = 0,\ \delta(P_2) > 0,\ p_2 \neq 0\}$$

Example

For the degree 4 Bezier surface $\Phi(u,v) = (u, v, h(u,v))$ with

$$h(u,v) = 116u^4v^4 - 200u^4v^3 + 108u^4v^2 - 24u^4v - 312u^3v^4 + 592u^3$$
$$+ 324u^2v^2 - 72u^2v - 56uv^4 + 112uv^3 - 72uv^2 + 16uv.$$



P is a bivariate polynomial

- total degree 84,
- degree 43 in u and v,
- 1907 terms,
- coefficients with up to 53 digits.

Outline

Problem statement

Find a topological graph of the ridge curve $P = 0$.
Classical method (Cylindrical Algebraic Decomposition)

1. Compute v-coordinates of singular and critical points: α_i
   Assume generic position
2. Compute intersection points between the curve and the line $v = \alpha_i$
   Compute with polynomial with *algebraic coefficients*
3. Connect points from fibers.

Our solution

1. Locate singular and critical points in 2D
   no generic position assumption
2. Compute regular intersection points between the curve and the fiber of singular and critical points
   Compute with polynomial with *rational coefficients*
3. Use the specific geometry of the ridge curve.
   We need to know how many branches of the curve pass throught each singular point.
4. Connect points from fibers.

Algebraic tools

1. Univariate root isolation for polynomial with rational coefficients.
2. Solve zero dimentional systems I with Rational Univariate Representation (RUR).
   - Recast the problem to an univariate one with rational functions.
   - Let t be a separating polynomial and f_t the characteristic polynomial of the multiplication by t in the algebra $\mathbb{Q}[X_1, \ldots, X_n]/I$

$$\begin{array}{ccc} V(I)(\cap\mathbb{R}^n) & \approx & V(f_t)(\cap\mathbb{R}) \\ \alpha = (\alpha_1, \ldots, \alpha_n) & \rightarrow & t(\alpha) \\ \left(\frac{g_{t,X_1}(t(\alpha))}{g_{t,1}(t(\alpha))}, \ldots, \frac{g_{t,X_n}(t(\alpha))}{g_{t,1}(t(\alpha))}\right) & \leftarrow & t(\alpha) \end{array}$$

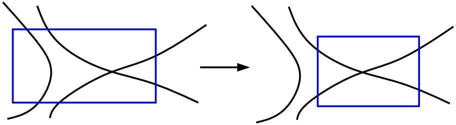
Outline

Step 1. Isolating study points

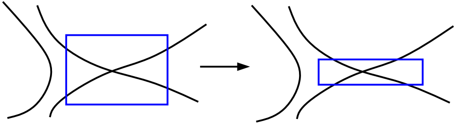
- Compute RUR of study points: 1-ridge umbilics, 3-ridge umbilics, purple points and critical points.
- Isolate study points in boxes $[u_i^1; u_i^2] \times [v_i^1; v_i^2]$, as small as desired.
- Identify study points with the same v-coordinate.

Step 2. Regularization of the study boxes

- Reduce a box until the right number of intersection points is reached wrt the study point type.

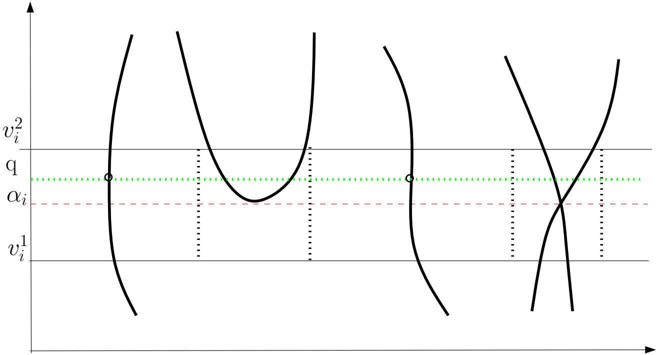


- Reduce to compute the number of branches connected from above and below.



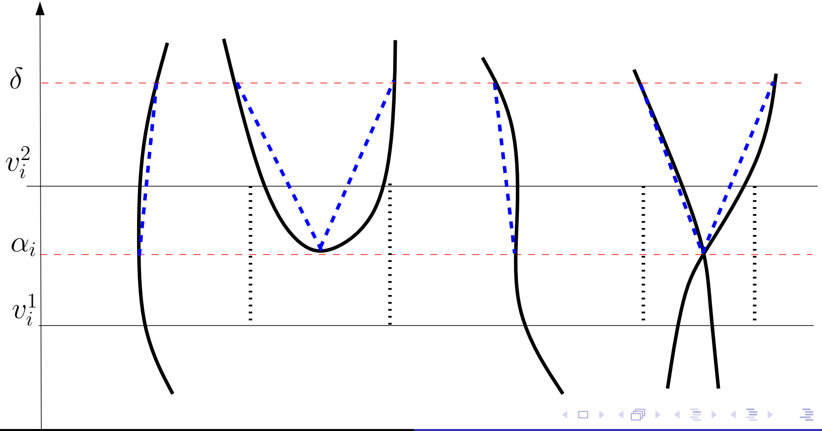
Step 3. Compute regular points in fibers

Outside study boxes, intersection between the curve and fibers of study points are regular points.
$\Longrightarrow$ Simple roots of the polynomial with rational coefficients $P(u, q)$ for any $q \in [v_i^1; v_i^2] \cap \mathbb{Q}$



Step 4. Perform connections

- Add intermediate fibers.
- One-to-one connection of points with multiplicity of branches.



Example: degree 4 Bezier surface

- Computation with the softwares FGB and RS.
- Domain of study $\mathcal{D} = [0,1] \times [0,1]$.

System	# of roots $\in \mathbb{C}$	# of roots $\in \mathbb{R}$	# of real roots $\in \mathcal{D}$
S_u	160	16	8
S_p	1068	31	17
S_c	1432	44	19

Example: degree 4 Bezier surface