Symmetry theorems for the newtonian 4- and 5-body problems with equal masses

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Abstract. We present a new proof of the algebraic part of a symmetry theorem for the central configurations of the newtonian planar 4-body problem with equal masses, using Gröbner bases. This approach is used to obtain a new symmetry theorem for the central configurations of the newtonian spatial 5-body problem with equal masses in the convex case. In fact we prove a more general statement of the theorem, valid for a class of potentials defined by functions with increasing and concave derivatives.

1 Introduction

The N-body problem is one of the most widely studied problems of Celestial mechanics. We are interested in a conceptually simple but important class of solutions, called central configurations. They are the only solutions of the N-body problem that can be computed analytically. As usual the study of a dynamical system begins with the study of its singularities. Central configurations are the singularities of the N-body problem and this is one of the reasons that justifies their significance. We are interested in the central configurations of the planar newtonian 4-body problem with equal masses and the central configurations of the spatial newtonian 5-body problem with equal masses.

A symmetry theorem for the planar newtonian 4-body problem with equal masses has been proved by A. Albouy. We give a simplified proof of the algebraic part of this theorem using Computer Algebra techniques. The most important fact about our proof is that it can be used to prove a new symmetry theorem for the spatial newtonian 5-body problem with equal masses, in the convex case. Indeed, we prove that every central configuration of the spatial newtonian 5-body problem with equal masses in the convex case, has at least one plane of symmetry. The proof uses several identities involving determinants and geometric properties of concave functions. These determinental identities have been obtained by Gröbner bases computations.

This result is of great importance for completing the classification of central configurations in the spatial newtonian 5-body problem with equal masses, that has been undertaken in [10].

2 Newtonian N-body problem with equal masses

In this section we will describe the general setting of the newtonian N-body problem with equal masses and we will give the equations of central configurations.

2.1 Notations

Consider n particles of masses $m_i, i = 1, \dots, n$ moving under their mutual gravitational attraction. Choose a point as the origin and denote by $r_i, i = 1, \dots, n$ the position vector of the i-th particle. Define the newtonian potential energy function as :

$$U = \sum_{1 \le i < j \le n} \frac{m_i m_j}{|| \, r_i - r_j \, ||}$$

where $|| \cdot ||$ is the euclidean norm.

More generally, one can define

$$U = \sum_{1 \le i < j \le n} m_i m_j \psi(|| \, r_i - r_j \, ||^2)$$

where ψ is a real-valued function. The newtonian case corresponds to $\psi(x) = x^{-1/2}$. The case of the logarithmic potential $\psi(x) = -\log x$ describes the central configurations of the vortex problem (see [2] and [11]).

The equations of motion (in the newtonian formulation) are :

$$m_i \, \ddot{r_i} \; = \; \frac{\partial U}{\partial r_i}, \quad i = 1, \dots, n. \tag{1}$$

Note that on the left-hand side we take the second derivative of the position vector w.r.t. time, that is the acceleration of the i-th particle, and that the right-hand side represents the total force exercised on the i-th particle by the other $n-1$.

The center of mass of the system is defined as :

$$r_c = \frac{m_1 r_1 + \cdots + m_n r_n}{m_1 + \cdots + m_n} \; .$$

The symmetry of the function U implies that $\ddot{r_c} = 0$, which means that r_c is a linear function of time. This is the principle of the conservation of linear momentum, which says that the center of mass moves in a straight line with uniform velocity. This allows us to shift the center of mass at the origin, without loss of generality and thus it remains to determine the motion relative to the center of mass. Using this and other similar considerations, like the conservation of energy and angular momentum, the order of the system is considerably reduced, but it is still quite difficult to study it.

2.2 Central Configurations

It is well-known that the equations of the N-body problem are in general not integrable. There exists a special important class of solutions which can be computed analytically. From the point of view of dynamical systems these are the singularities of the dynamical system in question. We give the mathematical definition of central configurations.

Let the center of mass of the system be at the origin of the coordinate axes. This means that we have $r_c = 0$. In this case, we have the following definition:

Definition 1. A configuration of the n particles is called a central configuration when the acceleration vector of each particle is a common scalar multiple of his position vector. In mathematical terms this means that there exists a scalar λ such that we have $\ddot{r}_i = \lambda r_i$ for $i = 1, \ldots, n$.

2.3 The equations of central configurations

In this paragraph we will give a characterization of central configurations by two groups of algebraic equations. The formulation of the equations of central configurations that we are going to present, is valid only for the (newtonian) N-body problem with equal masses in a euclidean space of dimension N-2. Due to homogeneity, the common value of the masses can be taken to be equal to one. The innovative idea of Dziobek (see [6]) was to use the mutual distances of the bodies as the coordinates to write the equations of central configurations. The resulting algebraic system is tractable by symbolic computation techniques after some preprocessing using linear algebra.

Let $r_{ij} = \| r_i - r_j \|$ for $1 \leq i < j \leq n$ and $s_{ij} = r_{ij}^2$. So s_{ij} is the square of the euclidean distance of the i-th particle from the j-th particle. There are $n(n-1)/2$ such mutual distances. Let $\psi(x)$ be a real function defining the potential. Following [1], we require that the function ψ' is strictly increasing ($\psi'' > 0$ and strictly concave ($\psi''' < 0$). These two hypotheses are satisfied for the newtonian potential ($\psi(x) = x^{-1/2}$) and for the logarithmic potential ($\psi(x) = -\log x$). Finally, let $\Delta_1, \ldots, \Delta_n$ be the oriented volumes of the n simplexes formed by the n bodies. These verify the relation:

$$\sum_{i=1}^{n} \Delta_i = 0. \tag{2}$$

Central configurations are then characterized by the following two groups of equations:

(A)	$\sum_{j \neq i} \Delta_j s_{ij} = X,$ for some X independent of i
(B)	there are two real numbers x and y such that $\psi'(s_{ij}) = x + y\, \Delta_i \Delta_j$

The group (A) equations are due to Albouy (see [1]). The group (B) equations for the central configurations of the planar newtonian 4-body problem (with arbitrary masses) appears in [6], where it is stated that the proof is based on Analytic Geometry considerations.

The general principles developed in [3] for the N-body problem in general, allow to prove both groups of equations using a construction in Linear Algebra.

3 Newtonian planar 4-body problem with equal masses

In this section we study the central configurations of the newtonian planar 4-body problem with equal masses.

3.1 The initial equations

In order to keep up with standard notation we put:

$$a = s_{12}, \quad b = s_{13}, \quad c = s_{14}, \quad d = s_{23}, \quad e = s_{24}, \quad f = s_{34} \tag{3}$$

and

$$\begin{array}{lll} A = \psi'(s_{12}), & B = \psi'(s_{13}), & C = \psi'(s_{14}), \\ D = \psi'(s_{23}), & E = \psi'(s_{24}), & F = \psi'(s_{34}). \end{array} \tag{4}$$

Finally, let $\Delta_1, \Delta_2, \Delta_3, \Delta_4$ be the oriented areas of the triangles $(2,3,4)$, $(4,3,1)$, $(1,2,4)$ and $(3,2,1)$ which verify

$$\Delta_1 + \Delta_2 + \Delta_3 + \Delta_4 = 0. \tag{5}$$

The group (A) gives the following equalities:

$$t_1 = t_2 = t_3 = t_4 \tag{6}$$

where

$$\begin{cases} t_1 = \Delta_2\ a + \Delta_3\ b + \Delta_4\ c \\ t_2 = \Delta_1\ a + \Delta_3\ d + \Delta_4\ e \\ t_3 = \Delta_1\ b + \Delta_2\ d + \Delta_4\ f \\ t_4 = \Delta_1\ c + \Delta_2\ e + \Delta_3\ f \end{cases}.$$

The group (B) gives the following equations:

$$\begin{cases} A = x + y\ \Delta_1\ \Delta_2,\ \ B = x + y\ \Delta_1\ \Delta_3,\ \ C = x + y\ \Delta_1\ \Delta_4, \\ D = x + y\ \Delta_2\ \Delta_3,\ \ E = x + y\ \Delta_2\ \Delta_4,\ \ F = x + y\ \Delta_3\ \Delta_4 \end{cases}. \tag{7}$$

3.2 Albouy's Symmetry theorem

The following theorem has been proved by A. Albouy in [1].

Theorem 2. *Every central configuration of the (newtonian) planar 4-body problem with equal masses has at least one symmetry.*

The proof of theorem (2) is done by contradiction and has two parts:

1. *The algebraic part* :
   where are obtained 4 equations of degree 2 as algebraic consequences of the initial equations.
2. *The geometric part* :
   where it is shown by geometric considerations and using the concavity of the derivative of the function ψ, that these 4 equations cannot be satisfied.

The algebraic part of the proof of theorem (2) is actually a way to eliminate the unknowns $x, y, \Delta_1, \Delta_2, \Delta_3, \Delta_4$. This is done in an elegant manner, in the following theorem:

Theorem 3. *The equations (5), (6) and (7) imply :*

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ A & B & C \end{vmatrix} + \begin{vmatrix} 1 & 1 & 1 \\ f & e & d \\ A & B & C \end{vmatrix} = 2 \begin{vmatrix} 1 & 1 & 1 \\ f & e & d \\ F & E & D \end{vmatrix} \tag{8}$$

$$\begin{vmatrix} 1 & 1 & 1 \\ a & e & d \\ A & E & D \end{vmatrix} + \begin{vmatrix} 1 & 1 & 1 \\ f & b & c \\ A & E & D \end{vmatrix} = 2 \begin{vmatrix} 1 & 1 & 1 \\ f & b & c \\ F & B & C \end{vmatrix} \tag{9}$$

$$\begin{vmatrix} 1 & 1 & 1 \\ f & b & d \\ F & B & D \end{vmatrix} + \begin{vmatrix} 1 & 1 & 1 \\ a & e & c \\ F & B & D \end{vmatrix} = 2 \begin{vmatrix} 1 & 1 & 1 \\ a & e & c \\ A & E & C \end{vmatrix} \tag{10}$$

$$\begin{vmatrix} 1 & 1 & 1 \\ f & e & c \\ F & E & C \end{vmatrix} + \begin{vmatrix} 1 & 1 & 1 \\ a & b & d \\ F & E & C \end{vmatrix} = 2 \begin{vmatrix} 1 & 1 & 1 \\ a & b & d \\ A & B & D \end{vmatrix} \tag{11}$$

We give now a new proof of theorem (3) using notions from Elimination theory and Gröbner bases theory.

Algebraic preliminaries In this paragraph we mention briefly some elements from Elimination theory and Gröbner bases theory which are needed in our proof of theorem (3). For a more detailed exposition and proofs of the theorems one can consult [4] and [5].

Let K be a field and consider the polynomial ring $K[x_1, \ldots, x_n]$ of polynomials in n variables over K. Let $f_1, \ldots, f_m$ be m polynomials in $K[x_1, \ldots, x_n]$ and consider the ideal $I = \langle f_1, \ldots, f_m \rangle$ generated by these polynomials.

Definition 4. An admissible order $>$ on $K[x_1,\ldots,x_n]$ is called a $k-$elimination order if

$$x_1^{a_1}\ldots x_n^{a_n} > x_{k+1}^{b_{k+1}}\ldots x_n^{b_n}$$

when $a_{i_0} > 0$ for some $i_0 \in \{1,\ldots,k\}$.

Remark 5. A lexicographical order is a $k-$elimination order for all k. A block order is defined using two admissible orders (see [4], p. 168). It is also a $k-$elimination order.

Definition 6. The kth elimination ideal I_k is the ideal of $K[x_{k+1},\ldots,x_n]$ defined by

$$I_k = I \cap K[x_{k+1},\ldots,x_n].$$

The fact that I_k is actually an ideal of $K[x_{k+1},\ldots,x_n]$ can be easily verified. The 0th elimination ideal is the ideal I himself. In more intuitive terms, the elements of I_k are all the consequences of the equations $f_1 = 0,\ldots,f_m = 0$, which do not contain the variables $x_1,\ldots,x_k$. In this terminology, eliminating $x_1,\ldots,x_k$ is equivalent to finding polynomials that belong to the kth elimination ideal I_k.

Gröbner bases provide a systematic way of finding elements of I_k using suitable orders on the variables. The following theorem shows how to use Gröbner bases to effectively compute a Gröbner basis of the kth elimination ideal I_k.

Theorem 7. *Let I be an ideal of $K[x_1,\ldots,x_n]$ and let G be a Gröbner basis of I with respect to a $k-$elimination order for k such that $0 \le k \le n$. Then the set*

$$G_k = G \cap K[x_{k+1},\ldots,x_n]$$

is a Gröbner basis of the kth elimination ideal I_k.

Symbolic proof of theorem (3) In order to prove theorem (3), we compute the Gröbner bases of equations (5), (6) and (7) with respect to three elimination orders that eliminate the variables $x,y,\Delta_1,\Delta_2,\Delta_3,\Delta_4$. Here are these three orders:

$$\mathcal{O}_1 = [x > y > \Delta_1 > \Delta_2 > \Delta_3 > \Delta_4 > A > B > C > D > E > F > a > b > c > d > e > f]$$

$$\mathcal{O}_2 = [x > y > \Delta_1 > \Delta_2 > \Delta_3 > \Delta_4 > A > B > C > D > E > F > d > e > f > a > b > c]$$

$$\mathcal{O}_3 = [x > y > \Delta_1 > \Delta_2 > \Delta_3 > \Delta_4 > A > B > C > D > E > F > f > e > d > c > b > a]$$

The Gröbner basis for the order $\mathcal{O}_1$ contains 44 elements. The following element has the smallest coefficients among the elements of degree two.

$$\begin{aligned} &aD - aE + bA + 2\,bC - bD - 2\,bF - cA - 2\,cB + cE + \\ &+2\,cF - dA + dE + eA - eD + 2\,fB - 2\,fC + fD - fE. \end{aligned} \tag{12}$$

The Gröbner basis for the order $\mathscr{O}_2$ contains 40 elements. The following element has the smallest coefficients among the elements of degree two.

$$\begin{aligned} &-aB-2\,aC+aD+2\,aE-bD+bF+2\,cA+cB-2\,cE \\ &-cF+dB-dF-2\,eA+2\,eC-eD+eF-fB+fD. \end{aligned} \tag{13}$$

The Gröbner basis for the order $\mathscr{O}_3$ contains 40 elements. The following element has the smallest coefficients among the elements of degree two.

$$\begin{aligned} &2\,aB+aC-2\,aD-aE-2\,bA-bC+2\,bD+bF+cE \\ &-cF+2\,dA-2\,dB+dE-dF-eC+eF+fC-fE \end{aligned} \tag{14}$$

These computations have been performed using the Gb program ([7]). The total computation time was less than a second.

A small program in MAPLE to detect linear combinations of equations (12), (13) and (14) with small coefficients, establishes that it suffices to add them to obtain such an equation:

$$\begin{aligned} &aB-aC-bA+bC+cA-cB+dA-dB+2\,dE \\ &-2\,dF-eA+eC-2\,eD+2\,eF+fB-fC+2\,fD-2\,fE \end{aligned} \tag{15}$$

We remark that in each of the equations (12), (13), (14) and (15) there are three monomials with a coefficient of 2 and three monomials with a coefficient of -2. Starting from these monomials we can easily construct the determinants appearing on the right-hand side of relations (8), (9), (10) and (11). The remaining 12 monomials can be easily shown to be the sums of determinants appearing on the left-hand side.

4 Newtonian spatial 5-body problem with equal masses

In this section we study the central configurations of the newtonian spatial 5-body problem with equal masses.

4.1 The initial equations

In this paragraph we maintain the notations introduced in section 2.3. Moreover we put $S_{ij}=\psi'(s_{ij})$. Let $\Delta_1,\Delta_2,\Delta_3,\Delta_4,\Delta_5$ be the oriented volumes of the five tetrahedra formed by the five bodies. They verify the equation

$$\Delta_1+\Delta_2+\Delta_3+\Delta_4+\Delta_5=0. \tag{16}$$

The group (A) gives the following equalities:

$$t_1=t_2=t_3=t_4=t_5 \tag{17}$$

where

$$\begin{cases} t_1 = \Delta_2\ s_{12} + \Delta_3\ s_{13} + \Delta_4\ s_{14} + \Delta_5\ s_{15} \\ t_2 = \Delta_1\ s_{12} + \Delta_3\ s_{23} + \Delta_4\ s_{24} + \Delta_5\ s_{25} \\ t_3 = \Delta_1\ s_{13} + \Delta_2\ s_{23} + \Delta_4\ s_{34} + \Delta_5\ s_{45} \\ t_4 = \Delta_1\ s_{14} + \Delta_2\ s_{24} + \Delta_3\ s_{34} + \Delta_5\ s_{45} \\ t_5 = \Delta_1\ s_{15} + \Delta_2\ s_{25} + \Delta_3\ s_{35} + \Delta_4\ s_{45} \end{cases}.$$

The group (B) gives the following equations:

$$S_{ij} = x + y\ \Delta_i\ \Delta_j \tag{18}$$

where

$$(i,j)\ \in\ \{(1,2),(1,3),(1,4),(1,5),(2,3),(2,4),(2,5),(3,4),(3,5),(4,5)\}.$$

4.2 The determinental equations

Let $S = \{1,\dots 5\}$. For integer indices i,j,k in S, let $l = \min(S \backslash \{i,j,k\})$ and $m = \max(S \backslash \{i,j,k\})$. (The indices l and m are uniquely determined by the indices i,j,k). Finally we note by E_{ijk} the equation:

$$\begin{vmatrix} 1 & 1 & 1 \\ s_{ij} & s_{ik} & s_{jk} \\ S_{lk} & S_{lj} & S_{li} \end{vmatrix} + \begin{vmatrix} 1 & 1 & 1 \\ s_{ij} & s_{ik} & s_{jk} \\ S_{mk} & S_{mj} & S_{mi} \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ s_{li} & s_{lj} & s_{lk} \\ S_{li} & S_{lj} & S_{lk} \end{vmatrix} + \begin{vmatrix} 1 & 1 & 1 \\ s_{mi} & s_{mj} & s_{mk} \\ S_{mi} & S_{mj} & S_{mk} \end{vmatrix} + 2 \begin{vmatrix} 1 & 1 & 1 \\ s_{ij} & s_{ik} & s_{jk} \\ S_{ij} & S_{ik} & S_{jk} \end{vmatrix}$$

Theorem 8. *The equations (16), (17) and (18) imply the equations:*

$$E_{ijk}\ \textit{for}\ 1 \le i < j < k \le 5. \tag{19}$$

Proof. Following the technique used in the 4-body problem we compute the Gröbner basis of equations (16), (17) and (18). We used a (DRL,DRL) block order, that is to say a DRL order on the 7 variables $x, y, \Delta_1, \Delta_2, \Delta_3, \Delta_4, \Delta_5$ and a DRL order on the 20 variables s_{ij}, S_{ij}. The computation has been performed with FGb ([7]) in less than 1 second. The Gröbner basis contains 6 equations of degree 2, $(e_1,\dots,e_6)$ in the variables s_{ij}, S_{ij}. We then use a small program in MAPLE to generate the 11^6 linear combinations $\sum_{i=1}^{6} \lambda_i\ e_i$ with $\mid \lambda_i \mid \le 5$. Keeping only the linear combinations with coefficients belonging to the set $\{-2,-1,1,2\}$, we obtain 716 homogeneous equations of degree two in s_{ij}, S_{ij}. By removing multiple elements we are left with 259 equations, of which only 20, have 30 monomials. A closer look to these 20 equations reveals that it suffices to keep only 10 of them, since the other 10 are just their opposites.

These 10 equations can be expressed in determinental form, just as in the case of the 4-body problem. □

Remark 9. The number of triplets of indices that appear in equations (19), is the number of ways of choosing 3 distinct elements form 5. Indeed, the binomial coefficient $\binom{5}{3}$ is equal to 10.

Remark 10. The ten equations E_{ijk} in theorem (8) appear in [2] where they are derived from more general considerations. We have established these equations independently and using exclusively equations (16), (17) and (18).

Remark 11. Equations (19) are not algebraically independent since there are only 6 equations of degree 2 in the Gröbner basis. This fact agrees with the theoretical prediction in [2].

Using equations (19), we will prove that in the convex case there are no central configurations without symmetry in the newtonian spatial 5-body problem with equal masses.

4.3 The symmetry theorem in the convex case

We call *convex configuration* every spatial configuration of the 5 bodies such that:

$$\Delta_1 \leq \Delta_2 \leq 0 \leq \Delta_3 \leq \Delta_4 \leq \Delta_5. \tag{20}$$

Using equations (19), we will prove that there is no solution of the initial system of equations that does not have symmetries.

Remark 12. In the rest of the paper we will assume that all the Δ_i are different than zero, because a zero Δ_i gives rise to a symmetric solution (see [10]).

We need a lemma on concave functions.

Lemma 13. *Let ϕ be a (strictly) concave function on* $\boldsymbol{R}$ *and $x > y > z$. Then we have*

$$\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ \phi(x) & \phi(y) & \phi(z) \end{vmatrix} > 0.$$

Proof. We put $\lambda = \dfrac{y-z}{x-z}$. According to the hypothesis $x > y > z$, we have that $\lambda \in (0,1)$. Moreover, we have that $y = z + \lambda(x-z) = \lambda x + (1-\lambda)z$. Since ϕ is a (strictly) concave function we have:

$$\phi(y) = \phi(\lambda x + (1-\lambda)z) > \lambda\phi(x) + (1-\lambda)\phi(z) = \lambda\phi(x) + \phi(z) - \lambda\phi(z)$$

and so we have

$$\phi(y) - \phi(z) > \lambda(\phi(x) - \phi(z))$$

which means that
$$\frac{\phi(y)-\phi(z)}{y-z} > \frac{\phi(x)-\phi(z)}{x-z}. \tag{21}$$

By developing the determinant in the statement of the theorem we have:
$$\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ \phi(x) & \phi(y) & \phi(z) \end{vmatrix} = \begin{vmatrix} x-z & y-z \\ \phi(x)-\phi(z) & \phi(y)-\phi(z) \end{vmatrix} > 0.$$

□

Remark 14. Geometrically, the determinant appearing in the statement of lemma (13) can be interpreted as the oriented area of the triangle formed by the three points $(x,\phi(x)),(y,\phi(y)),(z,\phi(z))$ on the graph of ϕ.

We now state and prove the symmetry theorem in the convex case.

Theorem 15. *Every central spatial convex configuration of five equal masses is symmetric.*

Proof. Suppose that the configuration is non symmetric, which implies that all inequalities in (20) are strict:
$$\Delta_1 < \Delta_2 < 0 < \Delta_3 < \Delta_4 < \Delta_5. \tag{22}$$

Without loss of generality we may assume that $y > 0$ (if $y < 0$ we obtain opposite inequalities).

Using (22) and equations (18) we can order the S_{ij}. For instance $S_{14} - S_{15} = y\Delta_1(\Delta_4 - \Delta_5)$ so that $S_{14} - S_{15} > 0$. In the same way we obtain:
$$\begin{gathered} S_{15} < S_{14} < S_{13} < S_{23} < S_{12}, \\ S_{14} < S_{24} < S_{34} < S_{35} < S_{45}, \\ S_{15} < S_{25} < S_{24} < S_{23} < S_{12}, \\ S_{25} < S_{35} \end{gathered} \tag{23}$$

We cannot compare directly S_{25} and S_{34}.

Since ψ' is a strictly increasing function ($\psi'' > 0$) and $S_{ij} = \psi'(s_{ij})$, we have the corresponding relations for s_{ij} :
$$\begin{gathered} s_{15} < s_{14} < s_{13} < s_{23} < s_{12}, \\ s_{14} < s_{24} < s_{34} < s_{35} < s_{45}, \\ s_{15} < s_{25} < s_{24} < s_{23} < s_{12}, \\ s_{25} < s_{35} \end{gathered} \tag{24}$$

We cannot compare directly s_{25} and s_{34}.

Relations (24) and (23) and lemma (13) can be used to decide the sign of the determinants appearing in equations (19).

In particular we will prove that equation E_{123} is impossible.

Indeed the two determinants on the left-hand side of E_{123} are strictly positive, because we have:

$$\begin{vmatrix} 1 & 1 & 1 \\ s_{12} & s_{13} & s_{23} \\ S_{34} & S_{24} & S_{14} \end{vmatrix} = \underbrace{(s_{12}-s_{23})}_{>0}\underbrace{(S_{24}-S_{14})}_{>0} + \underbrace{(s_{23}-s_{13})}_{>0}\underbrace{(S_{34}-S_{14})}_{>0} > 0$$

and

$$\begin{vmatrix} 1 & 1 & 1 \\ s_{12} & s_{13} & s_{23} \\ S_{35} & S_{25} & S_{15} \end{vmatrix} = \underbrace{(s_{12}-s_{23})}_{>0}\underbrace{(S_{25}-S_{15})}_{>0} + \underbrace{(s_{23}-s_{13})}_{>0}\underbrace{(S_{35}-S_{25})}_{>0} > 0.$$

Moreover, the three determinants on the right-hand side of E_{123} are strictly negative, because the function ψ' is (strictly) concave ($\psi''' < 0$) and by applying lemma (13) we have:

$$\begin{vmatrix} 1 & 1 & 1 \\ s_{14} & s_{24} & s_{34} \\ S_{14} & S_{24} & S_{34} \end{vmatrix} = - \begin{vmatrix} 1 & 1 & 1 \\ s_{34} & s_{24} & s_{14} \\ S_{34} & S_{24} & S_{14} \end{vmatrix} < 0 \text{ because } s_{34} > s_{24} > s_{14},$$

$$\begin{vmatrix} 1 & 1 & 1 \\ s_{15} & s_{25} & s_{35} \\ S_{15} & S_{25} & S_{35} \end{vmatrix} = - \begin{vmatrix} 1 & 1 & 1 \\ s_{35} & s_{25} & s_{15} \\ S_{35} & S_{25} & S_{15} \end{vmatrix} < 0 \text{ because } s_{35} > s_{25} > s_{15},$$

$$\begin{vmatrix} 1 & 1 & 1 \\ s_{12} & s_{13} & s_{23} \\ S_{12} & S_{13} & S_{23} \end{vmatrix} = - \begin{vmatrix} 1 & 1 & 1 \\ s_{12} & s_{23} & s_{13} \\ S_{12} & S_{23} & S_{13} \end{vmatrix} < 0 \text{ because } s_{12} > s_{23} > s_{13}.$$

Consequently, equation E_{123} is impossible and there is a contradiction. The hypothesis that the configuration is non symmetric is false. □

Corollary 16. *Every central spatial convex configuration of the newtonian 5-body problem with equal masses is symmetric.*

Theorem (15) can be proved under a weaker hypothesis than (22).

Proposition 17. *There is no central configuration such that* $\Delta_1 < \Delta_2 < 0 < \Delta_3 \leq \Delta_4 \leq \Delta_5$.

Proof. Using the hypothesis and equations (18) we see that between inequalities (23), some of them are no longer strict:

$$\begin{array}{c} S_{15} \leq S_{14} \leq S_{13} < S_{23} < S_{12}, \\ S_{14} < S_{24} < S_{34} \leq S_{35} \leq S_{45}, \\ S_{15} < S_{25} \leq S_{24} \leq S_{23} < S_{12}, \\ S_{25} < S_{35} \end{array}$$

Using the fact that ψ' is a strictly increasing function, we have the corresponding inequalities for the s_{ij}:

$$\begin{array}{c} s_{15} \leq s_{14} \leq s_{13} < s_{23} < s_{12}, \\ s_{14} < s_{24} < s_{34} \leq s_{35} \leq s_{45}, \\ s_{15} < s_{25} \leq s_{24} \leq s_{23} < s_{12}, \\ s_{25} < s_{35} \end{array}$$

The first 3×3 determinant of E_{123} remains strictly positive and there is still a contradiction.

Remark 18. It has been verified by direct Gröbner bases computations (using FGb) that the hypothesis

$$\Delta_1 = \Delta_2 \text{ and } \prod_{1<i<j<6} \Delta_i(\Delta_i - \Delta_j) \neq 0$$

gives rise to a contradiction, in the case of a logarithmic potential.

Theorem (15) is useful in the classification of central configuration types in the spatial newtonian 5-body problem with equal masses. In [10] and [9] one can find a detailed account of several symmetric central configuration types.

5 Conclusion

We used Symbolic Computation techniques to prove symmetry theorems for the newtonian planar 4-body problem with equal masses and the newtonian spatial 5-body problem with equal masses in the convex case. In the 4-body problem case, the symmetry theorem is enough to complete the classification of central configurations. In the 5-body problem case, the symmetry theorem needs to be extended for the non convex case, in order to complete the classification of central configurations. This will be done in a forthcoming paper.

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