Efficient computations of contiguity matrices for statistics and physics

Master 2 internship subject in Computer Algebra

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Context, scientific positioning

Exact methods for solving a polynomial system $f_1(x_1,\ldots,x_n) = \cdots = f_m(x_1,\ldots,x_n) = 0$, where $f_1,\ldots,f_m$ are defined over a field $\mathbb{K}$, rely on computing Gröbner bases of the ideal $I = \langle f_1,\ldots,f_m\rangle$. When the number of solutions is finite in $\bar{\mathbb{K}}^n$, the so-called *zero-dimensional* case, the typical framework to do so is to first compute a $\prec_{\mathrm{DRL}}$-Gröbner basis of I using Buchberger's algorithm [3] or Faugère's F_4 [4] or F_5 [5] algorithms. Then, a change of order algorithm, such as the seminal FGLM algorithm [6] or faster variants [2, 7, 8, 10] to recover the $\prec_{\mathrm{LEX}}$-Gröbner basis of I. This $\prec_{\mathrm{LEX}}$-Gröbner basis allows one to compute the coordinates of the solutions, like Gaussian elimination in the linear case, by solving, iteratively, univariate polynomials.

These change of order algorithms boil down to computing the multiplication matrices $M_{x_1},\ldots,M_{x_n}$, where M_{x_i} is the matrix of the linear map "multiplication by x_i" in the quotient algebra $\mathcal{A} = \mathbb{K}[x_1,\ldots,x_n]/I$ in the $\prec_{\mathrm{DRL}}$-monomial basis, and then use these matrices to compute the $\prec_{\mathrm{LEX}}$-Gröbner basis.

Assuming that the degree D of I is large compared to n, the cost of computing these multiplication matrices is $\tilde{O}(D^3)$ in [6]. Under a genericity assumption on the $\prec_{\mathrm{DRL}}$-Gröbner basis, called the *stability assumption*, this cost has been reduced to $\tilde{O}(D^\omega)$ in [10].

In statistics and physics [9], given a *likelihood function* $L = a_1^{x_1}\cdots a_p^{x_p} f_1^{-x_{p+1}}\ldots f_{n-p}^{-x_n}$ and the affine toric variety $X = \{a \in (\mathbb{C}^*)^p, f_1(a),\ldots,f_{n-p}(a) \neq 0\}$, the *twisted cohomology* can be computed by considering the *contiguity matrices* of a zero-dimensional ideal J in the ring of quasi-commutative polynomials $R = \mathbb{K}(\delta_1,\ldots,\delta_n)\langle x_1,\ldots,x_n\rangle$. In this ring, we have the following commutation rules

$$\begin{aligned}
\forall i, j,\ \delta_i\delta_j &= \delta_j\delta_i,\\
\forall i, j,\ x_i x_j &= x_j x_i,\\
\forall i \neq j,\ \delta_i x_j &= x_j\delta_i,\\
\forall i,\ (\delta_i + 1)x_i &= x_i\delta_i.
\end{aligned}$$

Then, these contiguity matrices are actually the multiplication matrices $M_{x_1},\ldots,M_{x_n}$ in the quotient algebra R/J.

As these quasi-commutative polynomials are over the field of fractions $\mathbb{K}(\delta_1,\ldots,\delta_n)$, the computation of these matrices is not as straightforward and efficient as in the commutative case.

Internship Objectives

The main objective of this internship is to design fast algorithms and high-level implementations, such as in Maple [1], for computing these contiguity matrices.

In the current implementations, the computations of these matrices are done directly with a naive arithmetic in $\mathbb{K}(\delta_1,\ldots,\delta_n)$. Therefore, eveluation-interpolation techniques will be investigated by the candidate. Likewise, when $\mathbb{K}$ has characteristic 0, Chinese remaindering techniques will be used simultaneously.

In some examples of [9], the input $\prec_{\mathrm{DRL}}$-Gröbner basis needed to build the multiplication matrices satisfies the generic assumption needed by [10] to speed the computation up. Thus, the student will investigate how this algorithm can be transposed to this quasi-commutative setting in order to bring the computation complexity down.

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Finally, in the commutative case, generically, only M_{x_n} is needed to compute the $\prec_{\text{LEX}}$-Gröbner basis. The student will investigate which contiguity matrices are really needed to recover the twisted cohomology of the examples of [9]. In particular, is the one of x_n enough?

Environment

This internship will take place in LIP6, a joint lab between Sorbonne Université and CNRS in Paris. The intern will join a dynamical and scientifically ambitious team which advises and co-advises Ph.D. students and postdoctoral researchers from France and many other countries (currently and recently: Italy, Germany, the Netherlands, Spain, UK, USA, Vietnam, etc.). The team organizes also a joint seminar with the MATHEXP team.

The intern will have access to office space, to all the necessary software, and to computing servers owned by the team.

This internship is particularly appropriate for students willing to pursue a Ph.D. after obtaining their Master degree and could lead to Ph.D. subject and collaboration with MPI in Leipzig, Germany or JKU in Linz, Austria.

Application

To apply, the candidate must send an email to jeremy.berthomieu@lip6.fr containing a resume, a summary of the courses followed by the candidate in Master 1 and 2, a motivation letter and a copy of the report on the grades obtained in Master 1.

References

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