Deciding reachability of the infimum of a multivariate polynomial *

Aurélien Greuet
Laboratoire de Mathématiques (LMV-UMR8100)
Université de Versailles-Saint-Quentin
45 avenue des États-unis
78035 Versailles Cedex, France
greuet@math.uvsq.fr

Mohab Safey El Din
UPMC, Université Paris 06
INRIA, Paris Rocquencourt Center
SALSA Project, LIP6/CNRS
UMR 7606, France
Mohab.Safey@lip6.fr

ABSTRACT

Let $f \in \mathbb{Q}[X_1, \ldots, X_n]$ be of degree D. Algorithms for solving the unconstrained global optimization problem $f^\star = \inf_{\mathbf{x}\in\mathbb{R}^n} f(\mathbf{x})$ are of first importance since this problem appears frequently in numerous applications in engineering sciences. This can be tackled by either designing appropriate quantifier elimination algorithms or by certifying lower bounds on $f^\star$ by means of sums of squares decompositions but there is no efficient algorithm for deciding if $f^\star$ is a minimum.

This paper is dedicated to this important problem. We design a probabilistic algorithm that decides, for a given f and the corresponding $f^\star$, if $f^\star$ is reached over $\mathbb{R}^n$ and computes a point $\mathbf{x}^\star \in \mathbb{R}^n$ such that $f(\mathbf{x}^\star) = f^\star$ if such a point exists. This algorithm makes use of algebraic elimination algorithms and real root isolation. If L is the length of a straight-line program evaluating f, algebraic elimination steps run in $O\big(\log(D-1)n^6(nL+n^4)\mathcal{U}\big((D-1)^{n+1}\big)^3\big)$ arithmetic operations in $\mathbb{Q}$ where $D = \deg(f)$ and $\mathcal{U}(x) = x\big(\log(x)\big)^2 \log\log(x)$. Experiments show its practical efficiency.

Categories and Subject Descriptors

I.1.2 [**Computing Methodologies**]: Symbolic and Algebraic Manipulation—*Algorithms*; G.1.6 [**Mathematics of computing**]: Numerical Analysis—*Optimization*

General Terms

Theory, algorithms

Keywords

Global optimization, polynomials

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1. INTRODUCTION

Motivations and problem statement. Consider the global optimization problem $f^\star = \inf_{\mathbf{x}\in\mathbb{R}^n} f(\mathbf{x})$ with $f \in \mathbb{Q}[X_1, \ldots, X_n]$. Solving these problems is of first importance since they occur frequently in engineering sciences (e.g. in systems theory or system identification [15]) or in the proof of some theorems (see [10, 18]). However, this infimum can be unreached: consider the polynomial $f(x, y) = (xy - 1)^2 + x^2$. As a sum of squares, $f \geq 0$. But f has only one critical value, $f(0, 0) = 1$, whereas $f(1/\ell, \ell)$ tends to 0 when ℓ tends to ∞. Thus, $f^\star = 0$, which is not a value taken by f.

Lower bounds on $f^\star$ can be computed via sums of squares decompositions which provide algebraic certificates of positivity [17, 22]. Following [21], this solving process can be improved when it is already known that $f^\star$ is a minimum but, in this framework, there is no given algorithm to test if $f^\star$ is a minimum. Global optimization problems can also be tackled via quantifier elimination (see [7, Chap. 14] or a dedicated algorithm in [25]). The algorithm given in [7, Chap. 14] allows to decide if $f^\star$ is a minimum within a complexity $D^{O(n)}$ but the complexity constant in the exponent is so large that it can't be used in practice. The algorithm given in [25] is much more efficient in practice but does not decide if $f^\star$ is a minimum. Our goal is to tackle this important problem.

To decide if $f^\star$ is a minimum, it is sufficient to decide if $f - f^\star = 0$ has real solutions. Recall that $f^\star$ is a real algebraic number whose degree may be large. Thus, one would prefer to decide if the real algebraic set defined by $\frac{\partial f}{\partial X_1} = \cdots = \frac{\partial f}{\partial X_n} = 0$ contains a point $\mathbf{x}$ such that $f(\mathbf{x}) = f^\star$. Difficulties arise when the aforementioned system generates an ideal which is not equidimensional and/or not radical and/or defines a non-smooth real algebraic set. This task could be tackled by using algorithms in [7, Chap. 14] running in time $D^{O(n)}$ (where $D = \deg(f)$) but, again, the complexity constants (in the exponent) are so large that these algorithms can't be used in practice. Other algorithms based on the so-called critical point method whose complexities are not known are described in [1, 27]. Another alternative consists in using Collins'Cylindrical Algebraic Decomposition [8].

We provide a probabilistic algorithm that decides if $f^\star$ is a minimum. When $f^\star$ is reached, it computes also a minimizer. It runs within $\widetilde{O}\left(n^6(nL+n^4)\mathcal{U}\big((D-1)^{n+1}\big)^3\right)$ arithmetic operations in $\mathbb{Q}$, where $D = \deg(f)$ and $\mathcal{U}(x) = x\big(\log(x)\big)^2 \log\log(x)$. Experiments show that it is practically more efficient of several orders of magnitude than other algorithms that can be used for a similar task.

Related works. Using computer algebra techniques to solve global optimization problems is an emerging trend. These problems are tackled using sums-of-squares decomposition to produce certificates of positivity [17, 22, 21, 14, 28] or quantifier elimination [7, 25]. The objects used in this paper are similar to those used in [14] where the existence of new algebraic certificates based on sums of squares are proved. Such objects are also used in [25] to design an efficient algorithm computing the global infimum of a multivariate polynomial over the reals.

Polar varieties are introduced in computer algebra in [2] to grab sample points in smooth equidimensional real algebraic sets (see also [3, 4, 26, 27, 5] and references therein). The interplay between properness properties of polar varieties to answer algorithmic questions in effective real algebraic geometry is introduced in [26]. An algorithm computing real regular points in singular real hypersurfaces is given in [6]. An algorithm for computing real points in each connected component of the real counterpart of a singular hypersurface is given in [24].

The paper is organized as follows. The algorithm is described in Section 2. Then in Section 3 we present the practical performances. A complexity analysis is done in Section 4. Finally, we give the correctness proof in Section 5.

2. DESCRIPTION OF THE ALGORITHM

We present now the algorithm. It takes as input a polynomial $f \in \mathbb{Q}[\mathbf{X}]$ bounded from below, P a univariate polynomial in $\mathbb{Q}[T]$ and I a real interval such that $f^\star = \inf_{\mathbf{x}\in\mathbb{R}^n} f(\mathbf{x})$ is the *unique root of P in I*. Such a polynomial can be obtained with the algorithm given in [25].

Given a matrix $\mathbf{A} \in GL_n(\mathbb{Q})$ and a polynomial f, we denote by $f^{\mathbf{A}}$ the polynomial $f^{\mathbf{A}}(\mathbf{X}) = f(\mathbf{AX})$.

Our algorithm is probabilistic: the correctness of the output depends on the choice of a random matrix $\mathbf{A}$. We prove in Section 5 that the bad choices of $\mathbf{A}$ (that is the choices of $\mathbf{A}$ such that the algorithm fails or returns a wrong result) are contained in a strict Zariski-closed subset of $GL_n(\mathbb{Q})$. Practically, this means that for a generic choice of $\mathbf{A}$, the algorithm returns a correct result.

To describe the algorithm, we need to introduce some classical subroutines in polynomial system solving solvers. A representation of an algebraic variety V means a finite set of polynomials generating V or a geometric resolution of V (see [13, 20]).

- DescribeCurve: takes as input a finite set of polynomials $\mathbf{F} \subset \mathbb{Q}[\mathbf{X}]$ and a polynomial $g \in \mathbb{Q}[\mathbf{X}]$ and returns a representation of $\overline{V(\mathbf{F}) - V(g)}^{\mathcal{Z}}$ if its dimension is at most 1, else it returns an error.
- Intersect: takes as input a representation of a variety V whose dimension is at most one and polynomials $g_1, \ldots, g_s$ and returns a representation of $V \cap V(g_1, \ldots, g_s)$.
- RealSolve: takes as input a rational parametrization of a 0-dimensional system V, f, the univariate polynomial P and an interval I isolating one real root $f^\star$ of P; it decides if there exists a real point $\mathbf{x} \in V$ such that $f(\mathbf{x}) = f^\star$ by returning a polynomial R for which $\mathbf{x}$ is a root and a box isolating $\mathbf{x}$ if such a point exists, else it returns false.

Note that DescribeCurve and Intersect can be implemented using any algebraic elimination technique (e.g. Gröbner basis, triangular sets, geometric resolution). The routine RealSolve relies exclusively on univariate evaluation and real root isolation.

IsReached

Input: $f \in \mathbb{Q}[X_1, \ldots, X_n]$ bounded below. A real interval I and $P \in \mathbb{Q}(T)$ encoding $f^\star = \inf_{\mathbf{x}\in\mathbb{R}^n} f(\mathbf{x})$.
Output: a boolean which equals `false` if $f^\star$ is not reached, a list L containing a polynomial and an interval encoding a point $\mathbf{x}$ such that $f(\mathbf{x}) = f^\star$ if $f^\star$ is reached.

1. choose randomly $\mathbf{A} \in GL_n(\mathbb{Q})$.
2. For $1 \le i \le n-1$ do
   a. $\mathbf{C}_{n-i+1} \leftarrow$ DescribeCurve$([\mathbf{X}_{\le i-1}, \frac{\partial f^{\mathbf{A}}}{\partial X_{i+1}}, \ldots, \frac{\partial f^{\mathbf{A}}}{\partial X_n}], \frac{\partial f^{\mathbf{A}}}{\partial X_1})$
   b. $\mathbf{F}_{n-i+1} \leftarrow$ Intersect$(\mathbf{C}_{n-i+1}, \frac{\partial f^{\mathbf{A}}}{\partial X_1}, \ldots, \frac{\partial f^{\mathbf{A}}}{\partial X_i})$;
   c. If $L \leftarrow$ RealSolve$(\mathbf{F}_{n-i+1}, f, P, I)$ is not empty return L.
3. a. $\mathbf{F}_1 \leftarrow [\mathbf{X}_{\le n-1}, \frac{\partial f^{\mathbf{A}}}{\partial X_1}, \ldots, \frac{\partial f^{\mathbf{A}}}{\partial X_n}]$;
   b. If $L \rightarrow$ RealSolve$(\mathbf{F}_1, f, P, I)$ is not empty return L.
4. return `false`.

3. PRACTICAL PERFORMANCES

We have implemented our algorithm using Gröbner basis engine FGb implemented in C by J.-C. Faugère [11]. We also used some results in [27] for computing the set of non-properness of a polynomial map to check properness assumptions required to apply Theorem 1.

Examples named **K1, K2, K3, K4, Vor1,** and **Vor2** are coming from applications and extracted from [18, 10]. We also consider a polynomial available at
`http://www.expmath.org/extra/9.2/sottile/SectIII.7.html`.

Computations have been performed on a PC under Scientific Linux OS release 5.5 on Intel(R) Xeon(R) CPUs E5420 at 2.50 GHz with 20.55G RAM. All these examples are global optimization problems arising in computer proofs of Theorems in computational geometry or related areas. In this context, exhibiting a minimizer is sometimes meaningful for the geometric phenomenon under study. None of these examples can be solved using the implementations QEPCAD, and REDLOG of Collins' Cylindrical Algebraic Decomposition within one week of computation, even when providing to CAD solvers the fact that $f^\star$ is an infimum. Also, our implementations of [1, 27] fail on these problems in less than one week.

The columns **D**, **n** and ♯Terms contain respectively the degree, the number of variables and the number of terms of the considered polynomial. As one can see, the implementation of our algorithm outperforms other implementations since one can solve previously unreachable problems.

	D	n	♯Terms	Time
Sot1	24	4	677	3 h.
Vor1	6	8	63	< 1 min.
Vor2	5	18	253	5 h.
K1	4	8	77	< 1 min.
K2	4	8	53	< 1 min.
K3	4	8	67	< 1 min.
K4	4	8	45	< 1 min.

4. COMPLEXITY RESULTS

Let $\mathbf{F} = (f_1, \ldots, f_s)$ and g in $\mathbb{Q}[X_1, \ldots, X_n]$ of degree bounded by d given by a straight-line program of size $\le L$.

We denote by δ_i^a the algebraic degree of $V(f_1, \ldots, f_i)$, δ^a the maximum of the previous δ_i^a and $\mathcal{U}(x) = x(\log x)^2 \log\log x$ (see [20]). We only estimate the complexities of steps relying on algebraic elimination algorithms (DescribeCurve and Intersect) using the following subroutines:

- GeometricSolve ([20]): given $\mathbf{F}$ and g as above, returns an equidimensional decomposition of $\overline{V(\mathbf{F}) \setminus V(g)}^{\mathcal{Z}}$, encoded by a set of irreducible lifting fibers, in time
$$O\left(s\log(d)n^4(nL+n^4)\mathcal{U}(d\delta^a)^3\right).$$
- LiftCurve ([20]): given an irreducible lifting fiber F of the above output, returns a rational parametrization of the lifted curve of F in time
$$O\left(s\log(d)n^4(nL+n^4)\mathcal{U}(d\delta^a)^2\right).$$
- OneDimensionalIntersect ([13] removing the Clean step): if $\langle\mathbf{F}\rangle$ is 1-dimensional, $\mathfrak{I}$ a geometric resolution of $\langle\mathbf{F}\rangle$ and a polynomial g, it returns a rational parametrization of $V(\mathfrak{I}+g)$ in time $O\left(n(L+n^2)\mathcal{U}(\delta^a)\mathcal{U}(d\delta^a)\right)$.

We deduce the complexity of the algebraic steps of our algorithm:

Proposition 1. *Let D be the degree of a polynomial $f \in \mathbb{Q}[X_1, \ldots, X_n]$ bounded below. There exists a probabilistic algorithm deciding whether the infimum of f is reached over the reals or not with a complexity within*
$$O\left(\log(D-1)n^6(nL+n^4)\mathcal{U}\left((D-1)^{n+1}\right)^3\right)$$
arithmetic operations in $\mathbb{Q}$.

Proof. We use LiftCurve with the output of GeometricSolve to obtain our DescribeCurve. Then we compute n geometric resolutions for n polynomials of degree at most $D-1$ using OneDimensionalIntersect as our Intersect routine. Using the Refined Bézout Theorem (see Theorem 12.3 and Example 12.3.1 page 223 in [12]) we can bound δ^a by $(D-1)^n$. Replacing s with n and δ^a with $(D-1)^n$ in the above complexity results and remarking that the costs of other steps are negligible ends the proof. $\square$

5. PROOF OF CORRECTNESS

Notations and basic definitions. For any set Y in a euclidean space, we denote by $\overline{Y}$ the closure of Y for the euclidean topology and by $\overline{Y}^{\mathcal{Z}}$ the Zariski closure of Y. Without more precision, a closed set means a closed set for the euclidean topology.

For $\mathbf{A} \in \mathrm{GL}_n(\mathbb{R})$ and $g \in \mathbb{Q}[X_1, \ldots, X_n]$, we write $g^{\mathbf{A}}(X) = g(\mathbf{A}X)$. Similarly, if $\mathbf{G} = (g_1, \ldots, g_s)$ is a finite subset of $\mathbb{Q}[X_1, \ldots, X_n]$, $\mathbf{G}^{\mathbf{A}} = (g_1^{\mathbf{A}}, \ldots, g_s^{\mathbf{A}})$. If I is an ideal of $\mathbb{Q}[X_1, \ldots, X_n]$, $I^{\mathbf{A}}$ is the ideal $\{g^{\mathbf{A}} \mid g \in I\}$.

We will consider objects, called *polar varieties* which are close to the ones already used in [2, 26] in the framework of non-singular algebraic sets. These objects are related to some projections. For $1 \le i \le n-1$, we will denote by Π_i the canonical projection $(x_1, \ldots, x_n) \to (x_1, \ldots, x_i)$ and by φ_i the canonical projection $(x_1, \ldots, x_n) \to x_i$.

For $g \in \mathbb{Q}[X_1, \ldots, X_n]$, $1 \le i \le n$ and $\mathbf{A} \in \mathrm{GL}_n(\mathbb{Q})$, we define

- $\mathbf{0}_{i-1}$ denotes the hyperplane in $\mathbb{C}^n$ defined by $X_1 = \cdots = X_{i-1} = 0$; by convention $\mathbf{0}_0$ denotes $\mathbb{C}^n$;
- $W_{n-i+1}^{\mathbf{A},g} = V\left(g^{\mathbf{A}}, \frac{\partial g^{\mathbf{A}}}{\partial X_{i+1}}, \ldots, \frac{\partial g^{\mathbf{A}}}{\partial X_n}\right)$;
- $\mathfrak{C}_{n-i+1}^{\mathbf{A},g} = \overline{\left(V\left(\frac{\partial g^{\mathbf{A}}}{\partial X_{i+1}}, \ldots, \frac{\partial g^{\mathbf{A}}}{\partial X_n}\right) \cap \mathbf{0}_{i-1}\right) \setminus V\left(\frac{\partial g^{\mathbf{A}}}{\partial X_1}\right)}^{\mathcal{Z}}$;
- $\mathscr{F}_{n-i+1}^{\mathbf{A},g} = \mathfrak{C}_{n-i+1}^{\mathbf{A},g} \cap V\left(\frac{\partial g^{\mathbf{A}}}{\partial X_1}, \ldots, \frac{\partial g^{\mathbf{A}}}{\partial X_i}\right)$.

When $i = n$, $W_{n-i+1}^{\mathbf{A},g} = V(g^{\mathbf{A}})$ and $\mathfrak{C}_1^{\mathbf{A},g} = \mathbf{0}_{n-1}$ by convention. The superscript g will be omitted when there is no ambiguity.

We will also make use of the notion of properness of a polynomial map. Given a polynomial map $\varphi : Y \to Z$ where Y and Z are euclidean spaces, we will say that φ is proper at $z \in Z$ if there exists a closed ball $\overline{B} \subset Z$ containing z such that $\varphi^{-1}(\overline{B})$ is closed and bounded. The map φ will be said to be proper if it is proper at any point in Z.

For $i = 0$, we denote by $W_0^{\mathbf{A}}$ the algebraic set
$$W_0^{\mathbf{A}} = \overline{\mathbb{V}\left(\frac{\partial f^{\mathbf{A}}}{\partial X_2}, \ldots, \frac{\partial f^{\mathbf{A}}}{\partial X_n}\right) \setminus \mathbb{V}\left(\frac{\partial f^{\mathbf{A}}}{\partial X_1}\right)}^{\mathcal{Z}}.$$
For $1 \le i \le n-2$, we denote by $W_i^{\mathbf{A}}$ the algebraic set
$$W_i^{\mathbf{A}} = \overline{\mathbb{V}\left(X_1, \ldots, X_i, \frac{\partial f^{\mathbf{A}}}{\partial X_{i+2}}, \ldots, \frac{\partial f^{\mathbf{A}}}{\partial X_n}\right) \setminus \mathbb{V}\left(\frac{\partial f^{\mathbf{A}}}{\partial X_{i+1}}\right)}^{\mathcal{Z}}.$$
At last for $i = n-1$, $W_{n-1}^{\mathbf{A}}$ stands for the algebraic set $\mathbb{V}(X_1, \ldots, X_{n-1})$.

Let $C^{\mathbf{A}}$ be a connected component of $\mathbb{V}(f^{\mathbf{A}}) \cap \mathbb{R}^n$. For $0 \le k \le n-2$, we denote by $C_k^{\mathbf{A}} = \mathbb{V}(X_1, \ldots, X_k) \cap C^{\mathbf{A}} \subset \mathbb{R}^n$ and by π_{k+1} the canonical projection
$$\begin{array}{rrcl} \pi_{k+1}: & \mathbb{R}^{n-k} & \longrightarrow & \mathbb{R} \\ & (x_{k+1}, \ldots, x_n) & \longmapsto & x_{k+1} \end{array}.$$

We will say that, given $\mathbf{A} \in \mathrm{GL}_n(\mathbb{C})$, property $\mathcal{P}(\mathbf{A})$ holds if, for $1 \le i \le n$, there exists algebraic sets $V_{n-i+1}^{\mathbf{A}} \subset V(f^{\mathbf{A}} - f^\star)$ such that for all connected component $C^{\mathbf{A}}$ of $V(f^{\mathbf{A}} - f^\star) \cap \mathbb{R}^n$

- the restriction of Π_{i-1} to $V_{n-i+1}^{\mathbf{A}}$ is proper;
- the boundary of $\Pi_i(C^{\mathbf{A}})$ is contained in $\Pi_i(C^{\mathbf{A}} \cap V_{n-i+1}^{\mathbf{A}})$.
- for $1 \le i \le n-1$, for all point $\mathbf{x}$ in a connected component $C^{\mathbf{A}} \subset V(f^{\mathbf{A}} - f^\star) \cap \mathbb{R}^n$ not belonging to $V_{n-i+1}^{\mathbf{A}}$, there exists a ball $\mathcal{B}$ containing $\mathbf{x}$ such that $\dim(\Pi_i(\mathcal{B} \cap C^{\mathbf{A}})) = i$.

Following [26], the properness property of Π_{i-1} implies that $\dim(V_{n-i+1}^{\mathbf{A}}) \le i-1$. We will prove in the sequel that $\mathcal{P}(\mathbf{A})$ holds for a generic choice of $\mathbf{A}$ by considering more general algebraic sets than polar varieties.

We will also say that property $\mathcal{Q}(\mathbf{A})$ holds if for all $1 \le i \le n$, $W_{n-i+1}^{\mathbf{A},f-\varepsilon} \cap \mathbf{0}_{i-1} \cap V\left(\frac{\partial f^{\mathbf{A}}}{\partial X_1}\right)$ (where ε is an infinitesimal) is empty.

Sketch of proof. Let $f \in \mathbb{Q}[X_1, \ldots, X_n]$ and $f^\star = \inf_{\mathbf{x} \in \mathbb{R}^n} f(\mathbf{x})$.

Theorem 1. *Suppose that $f^\star > -\infty$. Let $\mathbf{A} \in \mathrm{GL}_n(\mathbb{Q})$ be such that $\mathcal{P}(\mathbf{A})$ and $\mathcal{Q}(\mathbf{A})$ hold. Then, the union of the sets $\cup_{i=1}^n \mathscr{F}_{n-i+1}^{\mathbf{A}}$ meets every connected component of $V(f^{\mathbf{A}} - f^\star) \cap \mathbb{R}^n$.*

Under the assumption that $\mathcal{P}(\mathbf{A})$ and $\mathcal{Q}(\mathbf{A})$ hold, the above result allows us to reduce the problem of deciding the emptiness of $V(f^{\mathbf{A}} - f^\star) \cap \mathbb{R}^n$ to the one of deciding the emptiness of $\left(\cup_{i=1}^n \mathscr{F}_{n-i+1}^{\mathbf{A}}\right) \cap V(f^{\mathbf{A}} - f^\star) \cap \mathbb{R}^n$. Supposing that $\cup_{i=1}^n \mathscr{F}_{n-i+1}^{\mathbf{A}}$ has dimension 0, any solver for 0-dimensional polynomial system can be used to decide the emptiness of $V(f^{\mathbf{A}} - f^\star) \cap \mathbb{R}^n$.

Thus, Theorem 1 is algorithmically useful if it is easy to ensure that $\mathcal{P}(\mathbf{A})$ and $\mathcal{Q}(\mathbf{A})$ hold and if the set $\cup_{i=1}^n \mathscr{F}_{n-i+1}^{\mathbf{A}}$

have dimension 0. The result below ensures that $\mathcal{Q}(\mathbf{A})$ holds for a generic choice of $\mathbf{A}$ and that the set $\cup_{i=1}^{n}\mathscr{F}_{n-i+1}^{\mathbf{A}}$ has dimension at most 0.

THEOREM 2. *There exists a non-empty Zariski-open set $\mathscr{O}\subset \mathrm{GL}_n(\mathbb{C})$ s.t. for all $\mathbf{A}\in \mathrm{GL}_n(\mathbb{Q})\cap\mathscr{O}$ and $1\le i\le n$,*

- *the sets $\mathfrak{C}_{n-i+1}^{\mathbf{A}}$ have dimension at most* 1
- *the sets $\mathscr{F}_{n-i+1}^{\mathbf{A}}$ have dimension at most* 0*;*
- *the sets $W_{n-i+1}^{\mathbf{A},f-\varepsilon}\cap \mathbf{0}_{i-1}\cap V\left(\frac{\partial f^{\mathbf{A}}}{\partial X_1}\right)$ are empty*

Finally, it remains to show how to ensure $\mathcal{P}(\mathbf{A})$ in order to apply algorithmically Theorem 1. Again, the result below ensures $\mathcal{P}(\mathbf{A})$ holds if $\mathbf{A}$ is chosen generically.

THEOREM 3. *Let $V\subset\mathbb{C}^n$ be an algebraic variety of dimension d. There exists a non-empty Zariski-open set $\mathscr{O}\subset \mathrm{GL}_n(\mathbb{C})$ such that for all $\mathbf{A}\in\mathrm{GL}_n(\mathbb{Q})\cap\mathscr{O}$, and $1\le i\le d+1$, there exist algebraic sets $V_{n-i+1}^{\mathbf{A}}\subset V^{\mathbf{A}}$ such that for all connected component $C^{\mathbf{A}}$ of $V^{\mathbf{A}}\cap\mathbb{R}^n$*

(i) the restriction of Π_{i-1} to $V_{n-i+1}^{\mathbf{A}}$ is proper;
(ii) the boundary of $\Pi_i(C^{\mathbf{A}})$ is contained in $\Pi_i(C^{\mathbf{A}}\cap V_{n-i+1}^{\mathbf{A}})$.
(iii) for all point $\mathbf{x}$ in a connected component $C^{\mathbf{A}}\subset V(f^{\mathbf{A}}-f^\star)\cap\mathbb{R}^n$ not belonging to $V_{n-i+1}^{\mathbf{A}}$, there exists a ball $\mathcal{B}$ containing $\mathbf{x}$ such that $\dim(\Pi_i(B\cap C^{\mathbf{A}}))=i$.

The proof of Theorem 2 is widely inspired by [2]; the proof of Theorem 3 is inspired by [26] and a construction introduced in [27].

Proof of Theorem 1. We start with a Lemma which is a consequence of $\mathcal{P}(\mathbf{A})$.

LEMMA 1. *Suppose that $\mathcal{P}(\mathbf{A})$ holds and let $C^{\mathbf{A}}$ be a connected component of $V(f^{\mathbf{A}}-f^\star)\cap\mathbb{R}^n$. There exists $i_0\in\{1,\dots,n-1\}$ and a connected component $Z_{i_0}^{\mathbf{A}}$ of $C^{\mathbf{A}}\cap\mathbf{0}_{i_0-1}\cap\mathbb{R}^n$ such that $\varphi_{i_0}(Z_{i_0}^{\mathbf{A}})\neq\mathbb{R}$ and there exists $\mathbf{x}\in Z_{i_0}^{\mathbf{A}}$ such that $a=\varphi_{i_0}(\mathbf{x})$ lies in the boundary of $\varphi_{i_0}(Z_{i_0}^{\mathbf{A}})$. Moreover, there exists $r>0$ such that $B(\mathbf{x},r)\cap Z_{i_0}^{\mathbf{A}}\cap\varphi_{i_0}^{-1}(a)=\{\mathbf{x}\}$.*

PROOF. Since $\mathcal{P}(\mathbf{A})$ holds, the boundary of $\Pi_i(C^{\mathbf{A}})$ is contained in $\Pi_i(C^{\mathbf{A}}\cap V_{n-i+1}^{\mathbf{A}})$ for $1\le i\le n-1$. In particular, this implies that $\Pi_i(C^{\mathbf{A}})$ is closed.

Consider the largest $i_0\in\{1,\dots,n-1\}$ such that $C^{\mathbf{A}}\cap\mathbf{0}_{i_0-1}\neq\emptyset$ and $C^{\mathbf{A}}\cap\mathbf{0}_{i_0}=\emptyset$; whence $\varphi_{i_0}(C^{\mathbf{A}}\cap\mathbf{0}_{i_0-1})$ is a union of segments in the X_{i_0}-axis. This implies that there exists $\mathbf{y}$ in the intersection of $\Pi_{i_0}(C^{\mathbf{A}}\cap\mathbf{0}_{i_0-1})$ with the boundary of $\Pi_{i_0}(C^{\mathbf{A}})$. Note also that since $\Pi_{i_0}(C_{i_0}^{\mathbf{A}}\cap\mathbf{0}_{i_0-1})$ is a union of segments in the X_{i_0}-axis not containing the origin of this axis, one can choose $\mathbf{y}$ such that its X_{i_0}-coordinate belongs to the boundary of $\varphi_{i_0}(C^{\mathbf{A}}\cap\mathbf{0}_{i_0-1})$.

Since $\mathcal{P}(\mathbf{A})$ holds, the boundary of $\Pi_{i_0}(C^{\mathbf{A}})$ is itself contained in $\Pi_{i_0}(C^{\mathbf{A}}\cap V_{n-i_0+1}^{\mathbf{A}})$. Consequently, there exists $\mathbf{x}\in C^{\mathbf{A}}\cap V_{n-i_0+1}^{\mathbf{A}}$ such that $\Pi_{i_0}(\mathbf{x})=\mathbf{y}$. Moreover, since $\mathbf{y}\in\Pi_{i_0}(C^{\mathbf{A}}\cap\mathbf{0}_{i_0-1})$, $\mathbf{x}\in\mathbf{0}_{i_0-1}$. Let now $Z_{i_0}^{\mathbf{A}}$ be the connected component of $C^{\mathbf{A}}\cap\mathbf{0}_{i_0-1}\subset V(f^{\mathbf{A}}-f^\star)\cap\mathbf{0}_{i_0-1}\cap\mathbb{R}^n$ containing $\mathbf{x}$. Obviously, $Z_{i_0}^{\mathbf{A}}\subset C^{\mathbf{A}}$ is also a connected component of $V(f^{\mathbf{A}}-f^\star)\cap\mathbf{0}_{i_0-1}\cap\mathbb{R}^n$ and, by construction $\mathbf{x}\in Z_{i_0}^{\mathbf{A}}$ lies in $C^{\mathbf{A}}\cap\mathbf{0}_{i_0-1}\cap V_{n-i_0+1}^{\mathbf{A}}$. Since we have chosen $\mathbf{y}$ such that its X_{i_0}-coordinate lies in the boundary of $\varphi_{i_0}(C^{\mathbf{A}}\cap\mathbf{0}_{i_0-1})$, this implies that $\varphi_{i_0}(\mathbf{x})$ lies in the boundary of $\varphi_{i_0}(Z_{i_0}^{\mathbf{A}})$.

Let $a=\varphi_{i_0}(\mathbf{x})$. It remains to prove that there exists $r>0$ such that $\varphi_{i_0}^{-1}(a)\cap Z_{i_0}^{\mathbf{A}}\cap B(\mathbf{x},r)=\{\mathbf{x}\}$. To do that, we prove that it has dimension 0.

Suppose on the contrary that there exists a connected semi-algebraic set $\gamma\subset\varphi_{i_0}^{-1}(a)\cap Z_{i_0}^{\mathbf{A}}$ containing $\mathbf{x}$ such that $\gamma\neq\{\mathbf{x}\}$. Then, by construction, $\Pi_{i_0}(\gamma)=\Pi_{i_0}(\mathbf{x})$. Recall that $\Pi_{i_0}(\mathbf{x})$ lies in the boundary of $\Pi_{i_0}(C^{\mathbf{A}})$. Since $\mathcal{P}(\mathbf{A})$ holds $\Pi_{i_0}(\gamma)\in\Pi_{i_0}(V_{n-i_0+1}^{\mathbf{A}})$ and the restriction of Π_{i_0-1} to $V_{n-i_0+1}^{\mathbf{A}}$ is proper. This latter property implies that the restriction of Π_{i_0-1} to $V_{n-i_0+1}^{\mathbf{A}}$ is finite. Recall that $\mathbf{x}\in\mathbf{0}_{i_0-1}$, thus there exists $r>0$ small enough for which $\gamma\cap V_{n-i_0+1}^{\mathbf{A}}\cap B(\mathbf{x},r)=\{\mathbf{x}\}$. Since we have supposed that $\gamma\neq\{\mathbf{x}\}$, there exists $\mathbf{x}'\in\gamma\cap B(\mathbf{x},r)\setminus\{\mathbf{x}\}$; consequently $\mathbf{x}'\notin V_{n-i_0+1}^{\mathbf{A}}$. Since $\mathcal{P}(\mathbf{A})$ holds, there exists a ball $\mathcal{B}$ containing $\mathbf{x}'$ such that $\dim\left(\pi_{i_0}(\mathcal{B}\cap C^{\mathbf{A}})\right)=i_0$. This implies that $\pi_{i_0}(\mathbf{x}')=\pi_{i_0}(\mathbf{x})$ is not in the boundary of $\pi_{i_0}(C^{\mathbf{A}})$, a contradiction. □

Now, let $C^{\mathbf{A}}$ be a connected component of $V(f^{\mathbf{A}}-f^\star)\cap\mathbb{R}^n$. Remark that at all points of $C^{\mathbf{A}}$, all the partial derivatives of $f^{\mathbf{A}}$ vanish. By the above Lemma, there exists a connected component $Z_{i_0}^{\mathbf{A}}$ of $V(f^{\mathbf{A}}-f^\star)\cap\mathbf{0}_{i_0-1}\cap\mathbb{R}^n$ such that $Z_{i_0}^{\mathbf{A}}\subset C^{\mathbf{A}}$ and $\varphi_{i_0}(Z_{i_0}^{\mathbf{A}})\neq\mathbb{R}$ but is closed since $\mathcal{P}(\mathbf{A})$ holds. Note that this implies that all partial derivatives of $f^{\mathbf{A}}$ vanish at all points of $Z_{i_0}^{\mathbf{A}}$.

We prove below that $Z_{i_0}^{\mathbf{A}}$ has a non-empty intersection with $\mathfrak{C}_{n-i_0+1}^{\mathbf{A}}$. Since $Z_{i_0}^{\mathbf{A}}\subset C^{\mathbf{A}}$ and all the partial derivatives of $f^{\mathbf{A}}$ vanish at any point of $C^{\mathbf{A}}$, this will conclude the proof.

Let H be a real hyperplane orthogonal to the X_{i_0}-axis which does not meet $\varphi_{i_0}(Z_{i_0}^{\mathbf{A}})$. Because $\varphi_{i_0}(Z_{i_0}^{\mathbf{A}})$ is a closed and strict subset of $\mathbb{R}$, $\mathsf{dist}(Z_{i_0}^{\mathbf{A}},H)$ between $\varphi_{i_0}(Z_{i_0}^{\mathbf{A}})$ and H is reached at a point $\mathbf{x}^\star$ in $Z_{i_0}^{\mathbf{A}}$. By Lemma 1, one can also assume that there exists $r>0$ such that $\mathbf{x}^\star$ is the unique minimizer of $\mathsf{dist}(Z_{i_0}^{\mathbf{A}},H)$ in the ball $B(\mathbf{x}^\star,r)$. To finish the proof of Theorem 1, it is sufficient to prove the lemma below.

LEMMA 2. *The point $\mathbf{x}^\star\in C^{\mathbf{A}}$ belongs to $\mathfrak{C}_{n-i_0+1}$.*

Additionally, up to choosing a small $r>0$, one can suppose that $B(\mathbf{x}^\star,r)\cap\left(\left(V(f^{\mathbf{A}}-f^\star)\cap\mathbf{0}_{i_0-1}\right)\setminus Z_{i_0}^{\mathbf{A}}\right)=\emptyset$.

Roughly speaking, the idea of the proof is to consider algebraic sets $V(f^{\mathbf{A}}-(f^\star+e))\cap\mathbf{0}_{i_0-1}$ for small enough $e>0$ which are "deformations" of $V(f^{\mathbf{A}}-f^\star)\cap\mathbf{0}_{i_0-1}$, and exhibit a sequence of points $(\mathbf{x}_e)_e$ lying in $\mathfrak{C}_{n-i_0+1}\cap B(\mathbf{x}^\star,r)$ which converge to $\mathbf{x}^\star$ when $e\to 0$. To do that rigorously, we need to use materials about infinitesimals and Puiseux series that we introduce now.

Preliminaries on infinitesimals and Puiseux series. We denote by $\mathbb{R}\langle\varepsilon\rangle$ (resp. $\mathbb{C}\langle\varepsilon\rangle$) the real closed field (resp. algebraically closed field) of algebraic Puiseux series with coefficients in $\mathbb{R}$ (resp. $\mathbb{C}$), where ε is an infinitesimal. We will use the classical notions of bounded elements in $\mathbb{R}\langle\varepsilon\rangle^n$ (resp. $\mathbb{C}^n$) over $\mathbb{R}^n$ (resp. $\mathbb{C}^n$) and their limits. The limit of a bounded element $z\in\mathbb{R}\langle\varepsilon\rangle^n$ (resp. $z\in\mathbb{R}\langle\varepsilon\rangle^n$) will be denoted by $\lim_0(z)$. The ring homomorphism $\lim_0$ will also be used on sets of $\mathbb{R}\langle\varepsilon\rangle^n$ and $\mathbb{C}\langle\varepsilon\rangle^n$

Also for semi-algebraic sets $S\subset\mathbb{R}^n$ defined by a system of polynomial equations and inequalities, we will denote by $\mathsf{ext}(S,\mathbb{R}\langle\varepsilon\rangle)$ the solution set of the considered system in $\mathbb{R}\langle\varepsilon\rangle^n$. We refer to [7, Chap. 2.6] for precise statements of these notions.

Proof of Lemma 2. We simplify the notations by letting $\mathfrak{f}^{\mathbf{A}}=f^{\mathbf{A}}-f^\star$ and $V(\mathfrak{f}^{\mathbf{A}})_{i_0-1}=V(\mathfrak{f}^{\mathbf{A}})\cap\mathbf{0}_{i_0-1}$. By [23, Lemma 3.6], $V(\mathfrak{f}^{\mathbf{A}})_{i_0-1}\cap\mathbb{R}^n=\lim_0\left(V(\mathfrak{f}^{\mathbf{A}}-\varepsilon)\cup V(\mathfrak{f}^{\mathbf{A}}+\varepsilon)\right)\cap\mathbf{0}_{i_0-1}\cap\mathbb{R}^n=\lim_0(V(\mathfrak{f}^{\mathbf{A}}-\varepsilon)\cap\mathbf{0}_{i_0-1})\cap\mathbb{R}^n$. Then, there exists a connected component $C_\varepsilon^{\mathbf{A}}\subset\mathbb{R}\langle\varepsilon\rangle^n$ of $V(\mathfrak{f}^{\mathbf{A}}-\varepsilon)\cap\mathbf{0}_{i_0-1}\cap\mathbb{R}\langle\varepsilon\rangle^n$ such that $C_\varepsilon^{\mathbf{A}}$ contains a $\mathbf{x}_\varepsilon$ such that $\lim_0(\mathbf{x}_\varepsilon)=\mathbf{x}^\star$.

Thus $\mathsf{ext}(B(\mathbf{x}^\star,r),\mathbb{R}\langle\varepsilon\rangle)\cap C_\varepsilon^{\mathbf{A}} \neq \emptyset$. Since $\mathsf{ext}(B(\mathbf{x}^\star,r),\mathbb{R}\langle\varepsilon\rangle)$ is bounded over $\mathbb{R}$, $\mathsf{dist}(C_\varepsilon^{\mathbf{A}}\cap\mathsf{ext}(B(\mathbf{x}^\star,r),\mathbb{R}\langle\varepsilon\rangle),\mathsf{ext}(H,\mathbb{R}\langle\varepsilon\rangle))$ is also bounded over $\mathbb{R}$; let $\mathfrak{d}_0$ be its image by $\lim_0$. Since r has been chosen such that $B(\mathbf{x}^\star,r)$ has an empty intersection with all connected components of $V(\mathfrak{f}^{\mathbf{A}})_{i_0-1}\cap\mathbb{R}^n$ which are not $C^{\mathbf{A}}$, all points in $C_\varepsilon^{\mathbf{A}}\cap\mathsf{ext}(B(\mathbf{x}^\star,r)$ have their image by $\lim_0$ in $Z_{i_0}^{\mathbf{A}}$. This implies that $\mathfrak{d}_0=\mathsf{dist}(Z_{i_0}^{\mathbf{A}},H)$.

Let $S(\mathbf{x}^\star,r)\subset\mathbb{R}^n$ be the sphere centered at $\mathbf{x}^\star$ of radius r. Suppose for the moment that all points in

$$\mathsf{ext}(S(\mathbf{x}^\star,r),\mathbb{R}\langle\varepsilon\rangle)\cap C_\varepsilon^{\mathbf{A}}$$

don't minimize $\mathsf{dist}\left(C_\varepsilon^{\mathbf{A}}\cap\mathsf{ext}(B(\mathbf{x}^\star,r),\mathbb{R}\langle\varepsilon\rangle),\mathsf{ext}(H,\mathbb{R}\langle\varepsilon\rangle)\right)$. Thus $\mathsf{dist}\left(C_\varepsilon^{\mathbf{A}}\cap\mathsf{ext}(B(\mathbf{x}^\star,r),\mathbb{R}\langle\varepsilon\rangle),\mathsf{ext}(H,\mathbb{R}\langle\varepsilon\rangle)\right)$ is realized at a point $\mathbf{x}_\varepsilon^\star\in C_\varepsilon^{\mathbf{A}}$ lying in the interior of $\mathsf{ext}(B(\mathbf{x}^\star,r),\mathbb{R}\langle\varepsilon\rangle)$. Remark that this also implies that $\mathbf{x}_\varepsilon^\star$ is bounded over $\mathbb{R}$. Since $\mathfrak{d}_0=\mathsf{dist}(C^{\mathbf{A}},H)$ and $\mathbf{x}^\star$ is the unique point of $B(\mathbf{x}^\star,r)\cap C^{\mathbf{A}}$ realizing $\mathsf{dist}(C^{\mathbf{A}},H)$, $\mathbf{x}^\star=\lim_0(\mathbf{x}_\varepsilon^\star)$. Since $\mathfrak{C}_{n-i_0+1}^{\mathbf{A}}$ is defined by polynomials with coefficients in $\mathbb{Q}$, in order to conclude it remains to prove that $\mathbf{x}_\varepsilon^\star$ lies in $\mathsf{ext}(\mathfrak{C}_{n-i_0+1}^{\mathbf{A}},\mathbb{R}\langle\varepsilon\rangle)$.

Moreover, by the implicit function theorem [7, Chap. 3.5], $\frac{\partial f^{\mathbf{A}}}{\partial X_{i_0+1}},\ldots,\frac{\partial f^{\mathbf{A}}}{\partial X_n}$ vanish at $\mathbf{x}_\varepsilon^\star\in V(\mathfrak{f}^{\mathbf{A}}-\varepsilon)\cap\mathbf{0}_{i_0-1}$. By $\mathcal{Q}(\mathbf{A})$, this implies that $\frac{\partial f^{\mathbf{A}}}{\partial X_1}$ doesn't vanish at $\mathbf{x}_\varepsilon^\star$. Consequently, $\mathbf{x}_\varepsilon^\star$ lies in $\mathsf{ext}(\mathfrak{C}_{n-i_0+1}^{\mathbf{A}},\mathbb{R}\langle\varepsilon\rangle)$.

Now, we prove the claim announced above, that is that $\mathsf{dist}\left(C_\varepsilon^{\mathbf{A}}\cap\mathsf{ext}(B(\mathbf{x}^\star,r),\mathbb{R}\langle\varepsilon\rangle),\mathsf{ext}(H,\mathbb{R}\langle\varepsilon\rangle)\right)$ is not reached at $\mathsf{ext}(S(\mathbf{x}^\star,r),\mathbb{R}\langle\varepsilon\rangle)\cap C_\varepsilon^{\mathbf{A}}$. Suppose on the contrary that there exists $\mathbf{x}_\varepsilon\in\mathsf{ext}(S(\mathbf{x}^\star,r),\mathbb{R}\langle\varepsilon\rangle)\cap C_\varepsilon^{\mathbf{A}}$ which realizes this distance. Since $\mathbf{x}_\varepsilon\in\mathsf{ext}(S(\mathbf{x}^\star,r),\mathbb{R}\langle\varepsilon\rangle)$, $\mathbf{x}_\varepsilon$ is bounded over $\mathbb{R}$ and its image by $\lim_0$, that we will denote by $\mathbf{x}_0$, lies in $S(\mathbf{x}^\star,r)$ since $S(\mathbf{x}^\star,r)$ is defined by polynomials with coefficients in $\mathbb{R}$. Note also that $\mathbf{x}_0$ lies in $Z_{i_0}^{\mathbf{A}}$ since r has been chosen such that $B(\mathbf{x}^\star,r)\cap Z_{i_0}^{\mathbf{A}}$ has an empty intersection with all connected components of $V(\mathfrak{f}^{\mathbf{A}})_{i_0-1}\cap\mathbb{R}^n$ distinct from $Z_{i_0}^{\mathbf{A}}$. This is a contradiction since r has been also chosen small enough such that $\mathbf{x}^\star$ is the unique point $B(\mathbf{x}^\star,r)\cap Z_{i_0}^{\mathbf{A}}$ which realizes $\mathsf{dist}(Z_{i_0}^{\mathbf{A}},H)$. ☐

Proof of Theorem 2. We prove this result in three steps:

(i) there exists a non-empty Zariski open set $\mathscr{O}_1\subset\mathrm{GL}_n(\mathbb{C})$ such that for all $\mathbf{A}\in\mathrm{GL}_n(\mathbb{Q})\cap\mathscr{O}_1$, the Zariski-closure $\mathfrak{C}_{n-i+1}$ of $V\left(X_1,\ldots,X_{i-1},\frac{\partial f^{\mathbf{A}}}{\partial X_{i+1}},\ldots,\frac{\partial f^{\mathbf{A}}}{\partial X_n}\right)\setminus V\left(\frac{\partial f^{\mathbf{A}}}{\partial X_1}\right)$ has dimension at most 1;

(ii) $\forall\mathbf{A}\in\mathscr{O}_1$, $\mathfrak{C}_{n-i+1}\cap V\left(\frac{\partial f^{\mathbf{A}}}{\partial X_1}\right)$ has dimension at most 0;

(iii) there exists a non-empty Zariski open set $\mathscr{O}_2\subset\mathrm{GL}_n(\mathbb{C})$ such that for all $\mathbf{A}\in\mathrm{GL}_n(\mathbb{Q})\cap\mathscr{O}_2$, the n equations $X_1,\ldots,X_{i-1},\frac{\partial f^{\mathbf{A}}}{\partial X_1},\frac{\partial f^{\mathbf{A}}}{\partial X_{i+1}},\ldots,\frac{\partial f^{\mathbf{A}}}{\partial X_n}$ intersect transversely.

Then, writing $\mathscr{O}=\mathscr{O}_1\cap\mathscr{O}_2$ will give a non-empty Zariski open set satisfying the announced properties.

The most difficult step is step *(i)*. Its proof is widely inspired by [2, Proposition 3].

Proof of step (i). First, since $\mathfrak{C}_1^{\mathbf{A}}=\mathbf{0}_{n-1}$ is the line $\{X_1=\cdots=X_{n-1}=0\}$, the first point of the statement is obvious for $i=n$.

Let $1\le i\le n-1$. Remark that the differential $d_{\mathbf{x}}f^{\mathbf{A}}$ is the matrix product $\left[\frac{\partial f}{\partial X_1}(\mathbf{Ax})\ \cdots\ \frac{\partial f}{\partial X_n}(\mathbf{Ax})\right]\mathbf{A}$. If we denote by a_{ij} the coefficients of the matrix $\mathbf{A}$ then this product equals $\left[\sum_{k=1}^n a_{k1}\frac{\partial f}{\partial X_k}(\mathbf{Ax})\ \cdots\ \sum_{k=1}^n a_{k,n}\frac{\partial f}{\partial X_k}(\mathbf{Ax})\right]$ which means that for $1\le i\le n$, $\frac{\partial f^{\mathbf{A}}}{\partial X_i}(\mathbf{x})=\sum_{k=1}^n a_{ki}\frac{\partial f}{\partial X_k}(\mathbf{Ax})$. For $1\le i\le n-1$, consider the mapping $\Phi_i\colon\mathbb{C}^n\times\mathbb{C}^{n(n-i)}\to\mathbb{C}^{n-1}$ which maps a point $(\mathbf{x},a_{1,i+1},a_{2,i+1},\ldots,a_{n,i+1},a_{1,i+2},\ldots,a_{n,n})$, where $\mathbf{x}=(x_1,\ldots,x_n)$, to

$$\left(x_1,\ldots,x_{i-1},\sum_{k=1}^n a_{k,i+1}\frac{\partial f}{\partial X_k}(\mathbf{Ax}),\ldots,\sum_{k=1}^n a_{k,n}\frac{\partial f}{\partial X_k}(\mathbf{Ax})\right).$$

Thus the Jacobian matrix at a point $\alpha=(\mathbf{x},a_{1,i+1},\ldots)\in\mathbb{C}^n\times\mathbb{C}^{n(n-i)}$ is the evaluation at $\mathbf{Ax}$ of the matrix

$$\begin{bmatrix}\mathbf{I}_{i-1} & 0 & \cdots & & \cdots & & \cdots & & & & & \\ * & \cdots & * & \frac{\partial f}{\partial X_1} & \cdots & \frac{\partial f}{\partial X_n} & 0 & \cdots & \cdots & & & \\ \vdots & \cdots & \vdots & 0 & \cdots & 0 & \frac{\partial f}{\partial X_1} & \cdots & \frac{\partial f}{\partial X_n} & 0 & \cdots & 0 \\ \vdots & \cdots & \vdots & 0 & \cdots & 0 & \cdots & 0 & \ddots & 0 & \cdots & 0 \\ * & \cdots & * & 0 & \cdots & 0 & 0 & \cdots & 0 & \frac{\partial f}{\partial X_1} & \cdots & \frac{\partial f}{\partial X_n}\end{bmatrix},$$

where $\mathbf{I}_{i-1}$ is the identity matrix of size $i-1$.

Consider the Zariski-open set $\mathcal{V}^{\mathbf{A}}$ of all points in $\mathbb{C}^n$ such that at least one partial derivative of $f^{\mathbf{A}}$ does not vanish. Then we prove that the restriction of Φ_i to $\mathcal{V}^{\mathbf{A}}\times\mathbb{C}^{n(n-i)}$ is transverse to $(0,\ldots,0)\in\mathbb{C}^{n-1}$. Indeed, we consider a point $\alpha=(\mathbf{x},a_{1,i+1},\ldots,a_{nn})\in\mathcal{V}^{\mathbf{A}}\times\mathbb{C}^{n(n-i)}$ with $\Phi_i(\alpha)=0$.

Suppose that the Jacobian matrix Φ_i has not maximal rank at α. Then all the partial derivative in the matrix have to vanish. This implies that all the partial derivatives of $f^{\mathbf{A}}$ vanish too, which is impossible if $\alpha\in\mathcal{V}^{\mathbf{A}}\times\mathbb{C}^{n(n-i)}$. Thus the Jacobian matrix has maximal rank at α, which means that α is a regular point of Φ_i. This is true for all $\alpha\in\Phi_i^{-1}(0)\cap\left(\mathcal{V}^{\mathbf{A}}\times\mathbb{C}^{n(n-i)}\right)$ therefore the restriction of Φ_i is transverse to $(0,\ldots,0)$ as announced. Then by the Weak Transversality Theorem of Thom-Sard (see [9, Theorem 3.7.4 p. 79]), there exists a Zariski-open set $\mathscr{O}_1\subset GL_n(\mathbb{C})$ such that for all $\mathbf{A}\in GL_n(\mathbb{Q})\cap\mathscr{O}_1$, the restriction of Φ_i to $\mathcal{V}^{\mathbf{A}}\times\mathbb{C}^{n(n-i)}$ is transverse to $(0,\ldots,0)$. This means that for all $\mathbf{A}\in GL_n(\mathbb{Q})\cap\mathscr{O}_1$, the equations $\frac{\partial f^{\mathbf{A}}}{\partial X_{i+1}}=0,\ldots,\frac{\partial f^{\mathbf{A}}}{\partial X_n}=0$, $X_1=0,\ldots,X_{i-1}=0$ intersect transversely at any of their common solutions which are in $\mathcal{V}^{\mathbf{A}}$. In particular this is true for the solutions in $\left\{\frac{\partial f^{\mathbf{A}}}{\partial X_1}\neq 0\right\}\subset\mathcal{V}^{\mathbf{A}}$. Finally, this means that if the algebraic variety

$$\overline{V\left(X_1,\ldots,X_{i-1},\frac{\partial f^{\mathbf{A}}}{\partial X_{i+1}},\ldots,\frac{\partial f^{\mathbf{A}}}{\partial X_n}\right)\setminus V\left(\frac{\partial f^{\mathbf{A}}}{\partial X_1}\right)}^{\mathcal{Z}},$$

which is precisely $\mathfrak{C}_{n-i+1}^{\mathbf{A}}$, has dimension one or is empty. ☐

Proof of step (ii). Let $\mathscr{O}_1$ be the Zariski-closed open set given in the previous proof. Let $\mathbf{A}\in GL_n(\mathbb{Q})\cap\mathscr{O}_1$ and let $i\in\{1,\ldots,n-1\}$. Then according to step (i), $\mathfrak{C}_{n-i+1}^{\mathbf{A}}$ has dimension at most one. Assume that $\mathfrak{C}_{n-i+1}^{\mathbf{A}}\cap V\left(\frac{\partial f^{\mathbf{A}}}{\partial X_1}\right)$ is nonempty. Then by definition, $\mathfrak{C}_{n-i+1}^{\mathbf{A}}$ is not included in $V\left(\frac{\partial f^{\mathbf{A}}}{\partial X_1}\right)$. By Krull's Principal Ideal Theorem [19, Cor. 3.2 p. 131], we deduce that $\dim\left(\mathfrak{C}_{n-i+1}^{\mathbf{A}}\cap V\left(\frac{\partial f^{\mathbf{A}}}{\partial X_1}\right)\right)=\dim(\mathfrak{C}_{n-i+1}^{\mathbf{A}})-1\le 0$. Then, $\mathscr{F}_{n-i+1}^{\mathbf{A}}=\mathfrak{C}_{n-i+1}^{\mathbf{A}}\cap V\left(\frac{\partial f^{\mathbf{A}}}{\partial X_1}\right)\cap V\left(\frac{\partial f^{\mathbf{A}}}{\partial X_2},\ldots,\frac{\partial f^{\mathbf{A}}}{\partial X_n}\right)$ has also dimension less than 0.

Let $i=n$ and assume that $\mathscr{F}_1\neq\varnothing$. Then if $\mathbf{A}$ is generic enough, it is clear that there exists $k\in\{1,\ldots,n\}$ such that $\mathfrak{C}_1^{\mathbf{A}}=\mathbf{0}_{n-1}$ in not contained in $V\left(\frac{\partial f^{\mathbf{A}}}{\partial X_k}\right)$. As above, by Krull's Principal Ideal Theorem we deduce that $\dim\left(\mathfrak{C}_1^{\mathbf{A}} \cap V\left(\frac{\partial f^{\mathbf{A}}}{\partial X_k}\right)\right) = \dim(\mathfrak{C}_1^{\mathbf{A}}) - 1 \leq 0$, which implies that $\mathscr{F}_1 = \mathfrak{C}_1 \cap V\left(\frac{\partial f^{\mathbf{A}}}{\partial X_1}, \ldots, \frac{\partial f^{\mathbf{A}}}{\partial X_n}\right)$ has dimension ≤ 0. $\square$

Proof of step (iii). For $1 \leq i \leq n$, consider the mapping Ψ_i: $\mathbb{C}^n \times \mathbb{C}^{n(n-i+1)} \to \mathbb{C}^{n+1}$ which maps a point

$$(\mathbf{x}, a_{1,1}, \ldots, a_{n,1}, a_{1,i+1}, a_{2,i+1}, \ldots, a_{n,i+1}, a_{1,i+2}, \ldots, a_{n,n}),$$

where $\mathbf{x} \in \mathbb{C}^n$ to $\left(x_1, \ldots, x_{i-1}, \frac{\partial f^{\mathbf{A}}}{\partial X_1}, \frac{\partial f^{\mathbf{A}}}{\partial X_{i+1}}, \ldots, \frac{\partial f^{\mathbf{A}}}{\partial X_n}\right)$.

Consider the Zariski-open set $\mathcal{V}^{\mathbf{A}}$ of all points in $\mathbb{C}^n$ such that at least one partial derivative of $f^{\mathbf{A}}$ does not vanish. As above, we prove that the restriction of Ψ_i to $\mathcal{V}^{\mathbf{A}} \times \mathbb{C}^{n(n-i+1)}$ is transverse to $(0, \ldots, 0) \in \mathbb{C}^n$: we consider a point $\beta = (\mathbf{x}, a_{1,i+1}, \ldots, a_{nn}) \in \mathcal{V}^{\mathbf{A}} \times \mathbb{C}^{n(n-i+1)}$ with $\Phi_i(\beta) = 0$. Because $\mathbf{x} \in \mathcal{V}^{\mathbf{A}}$, at least one partial derivative of $f^{\mathbf{A}}$ does not vanish at $\mathbf{x}$, which means that at least one partial derivative of f does not vanish at $\mathbf{Ax}$. Thus the shape of the Jacobian matrix of Ψ_i is such that it has maximal rank at β and β is a regular point of Ψ_i. Therefore the restriction of Ψ_i is transverse to $(0, \ldots, 0)$. Then by the Weak Transversality Theorem of Thom-Sard, there exists a Zariski-open set $\mathscr{O}_2 \subset GL_n(\mathbb{C})$ such that for all $\mathbf{A} \in GL_n(\mathbb{Q}) \cap \mathscr{O}_2$, the restriction of Ψ_i to $\mathcal{V}^{\mathbf{A}} \times \mathbb{C}^{n(n-i+1)}$ is transverse to $(0, \ldots, 0)$. This means that for all $\mathbf{A} \in GL_n(\mathbb{Q}) \cap \mathscr{O}_2$, the equations $\frac{\partial f^{\mathbf{A}}}{\partial X_1} = 0$, $\frac{\partial f^{\mathbf{A}}}{\partial X_{i+1}} = 0$, $\ldots$, $\frac{\partial f^{\mathbf{A}}}{\partial X_n} = 0$, $X_1 = 0$, $\ldots$, $X_{i-1} = 0$ intersect transversely at any of their common solutions which are in $\mathcal{V}^{\mathbf{A}}$. Because ε is an infinitesimal, the variety $V\left(f^{\mathbf{A}} - \varepsilon\right)$ is smooth, thus is a subset of $\mathcal{V}^{\mathbf{A}}$. Then $W_{n-i+1}^{\mathbf{A}, f-\varepsilon} \cap \mathbf{0}_{i-1} \cap V\left(\frac{\partial f^{\mathbf{A}}}{\partial X_1}\right)$, that is the intersection of $V\left(X_1, \ldots, X_{i-1}, \frac{\partial f^{\mathbf{A}}}{\partial X_1}, \frac{\partial f^{\mathbf{A}}}{\partial X_{i+1}}, \ldots, \frac{\partial f^{\mathbf{A}}}{\partial X_n}\right)$ with the hypersurface $V(f^{\mathbf{A}} - \varepsilon)$, is empty. $\square$

Proof of Theorem 3. We start with the third point.

LEMMA 3. *Let $C^{\mathbf{A}}$ be a connected component of $V(f^{\mathbf{A}} - f^\star)$. For all $i \in \{1, \ldots, n-1\}$, for all $\mathbf{x} \in C^{\mathbf{A}}$ such that $\mathbf{x} \notin V_{n-i+1}^{\mathbf{A}}$, there exists a ball $\mathcal{B}_i$ containing $\mathbf{x}$ such that $\dim\left(\pi_i(\mathcal{B}_i \cap C^{\mathbf{A}})\right) = i$.*

PROOF. For $i = n-1$, let $\mathbf{x} = (x_1, \ldots, x_n) \notin V_2$. By definition of V_2, $\mathbf{x}$ is in the $n-1$-equidimensional component of $V(f - f^\star)$ and is not a critical point of the restriction to C of π_{n-1}. Then using the implicit functions theorem, there exists a ball $\mathcal{B}_{n-1}$ centered on $\mathbf{x}$ and a continuously differentiable function ϕ such that for every $\mathbf{y} \in \mathcal{B}_{n-1}$, $\mathbf{y} \in C$ iff $\mathbf{y} = (y_1, \ldots, y_{n-1}, \phi(y_1, \ldots, y_{n-1}))$. Then the image of $\mathcal{B}_{n-1} \cap C$ by π_{n-1} has dimension $n-1$.

Let $i \in \{1, \ldots, n-1\}$ and assume that for all $\mathbf{x} \notin V_{n-i}$, there exists a ball $\mathcal{B}_{n-i}$ centered on $\mathbf{x}$ such that the projection $\pi_{i+1}\left(\mathcal{B}_{n-i} \cap C\right)$ has dimension $i+1$. Let us show that this implies that for all $\mathbf{x} \notin V_{n-i+1}$, there exists a ball $\mathcal{B}_{n-i+1}$ centered on $\mathbf{x}$ such that $\dim\left(\pi_i\left(\mathcal{B}_{n-i+1} \cap C\right)\right) = i$.

Let $\mathbf{x} \notin V_{n-i+1}$. If $\mathbf{x} \notin V_{n-i}$ then by assumption, there exists a ball $\mathcal{B}_{n-i}$ centered on $\mathbf{x}$ such that the projection $\pi_{i+1}\left(\mathcal{B}_{n-i} \cap C\right)$ has dimension $i+1$. It is clear that for all $j \leq i+1$, $\pi_j\left(\mathcal{B}_{n-i} \cap C\right)$ has dimension j. In particular for $j = i$, the result is proved.

Else, $\mathbf{x} \in V_{n-i}$ and $\mathbf{x} \notin V_{n-i+1}$. By definition of V_{n-i+1} and V_{n-i}, $\mathbf{x}$ belongs to the i-equidimensional component of V_{n-i}. Then this component is locally defined by $n-i$ equations. Moreover, $\mathbf{x}$ is not in singular locus of $\mathbb{V}_{n-i}$ and not in the critical locus of the projection π_i. This means that the Jacobian of the $n-i$ equations defining locally the i-equidimensional component of V_{n-i} with respect to the variables $x_{i+1}, \ldots, x_n$ is invertible. Then the implicit functions theorem applies and ensures that there exists a ball $\mathcal{B}_{n-i+1}$ centered on $\mathbf{x}$ and a continuously differentiable function ϕ such that for every $\mathbf{y} \in \mathcal{B}_{n-i+1}$, $\mathbf{y} \in C$ iff $\mathbf{y} = (y_1, \ldots, y_i, \phi(y_1, \ldots, y_i))$. Then the image of $\mathcal{B}_{n-i+1} \cap C$ by π_{n-i+1} has dimension i. $\square$

Then we give the intuition of the proof of the first two points. It consists by constructing recursively $V_{n-i+1}^{\mathbf{A}}$ from $V_{n-i}^{\mathbf{A}}$ with $V_{n-d}^{\mathbf{A}} = V^{\mathbf{A}}$. Suppose that we have found $\mathbf{A}$ such that properties *(i)* and *(ii)* are satisfied by $V_{n-i}^{\mathbf{A}}$. Then, we need to construct $V_{n-i+1}^{\mathbf{A}}$ in such a way that the boundary of $\Pi_i(C^{\mathbf{A}})$ is contained in $\Pi_i(C^{\mathbf{A}} \cap V_{n-i+1}^{\mathbf{A}})$. We will see that the implicit function theorem and the properness property of the restriction of Π_i to $V_{n-i}^{\mathbf{A}}$ enables us to choose $V_{n-i+1}^{\mathbf{A}}$ as the union of

- the j-equidimensional components of $V_{n-i}^{\mathbf{A}}$ for $1 \leq j \leq i-1$
- the singular locus of the i-equidimensional component of $V_{n-i}^{\mathbf{A}}$.
- the critical locus of the restriction of Π_i to the i-equidimensional component of $V_{n-i}^{\mathbf{A}}$;

Nevertheless, for this matrix $\mathbf{A}$, the restriction of Π_{i-1} to $V_{n-i+1}^{\mathbf{A}}$ may not be proper. Then, a generic change of variables on the coordinates $X_1, \ldots, X_i$ will not alter $V_{n-i+1}^{\mathbf{A}}$ but will restore the properness property of Π_{i-1}.

Our proof is widely inspired by the one of [26, Theorem 1 and Proposition 2] which state a similar result when V is smooth and equidimensional.

As in [26], to obtain the existence of the Zariski-open set $\mathscr{O}$, we need to adopt an algebraic viewpoint.

Strategy of proof. To adopt this algebraic viewpoint, we consider a finite family $\mathbf{F} \subset \mathbb{Q}[X_1, \ldots, X_n]$ generating the ideal associated to V which has dimension d. As in Section 2, $\mathbf{X}_{\leq i}$ denotes $X_1, \ldots, X_i$ for $1 \leq i \leq n$ and $\mathbf{X}_{\geq i}$ denotes $X_i, \ldots, X_n$.

Let $\mathfrak{A}$ be a $n \times n$ matrix whose entries are new indeterminates $(\mathfrak{A}_{i,j})_{1 \leq i,j \leq n}$. Define $f^{\mathfrak{A}} \in \mathbb{Q}(\mathfrak{A}_{i,j})[\mathbf{X}]$ as $f^{\mathfrak{A}} = f(\mathfrak{A}\mathbf{X})$. Thus, $\mathbf{F}^{\mathfrak{A}}$ denotes the set obtained by performing the change of variables $\mathfrak{A}$ on all polynomials in $\mathbf{F}$. This notation is also used for polynomial ideals. We will also denote by $\mathfrak{k}$ an algebraic closure of $\mathbb{Q}(\mathfrak{A}_{i,j})$. Finally, given an ideal I in $k[X_1, \ldots, X_n]$ where k is a field, we denote by $\mathbf{G}(I)$ a finite set of generators (e.g. a Gröbner basis) of I.

Our construction is recursive. We start by defining $\Delta_{n-d}^{\mathfrak{A}} = \langle \mathbf{F}^{\mathfrak{A}} \rangle \subset \mathbb{Q}(\mathfrak{A}_{i,j})[\mathbf{X}]$. Remark that $\dim(\Delta_{n-d}^{\mathfrak{A}}) = d$ and $\Delta_{n-d}^{\mathfrak{A}}$ is radical (since $\langle \mathbf{F} \rangle$ is so). Then, for $1 \leq i \leq d$, we denote by $\Delta_{n-d,n-i}^{\mathfrak{A}}$ the intersection of the prime ideals of co-dimension $n-i$ associated to $\Delta_{n-d}^{\mathfrak{A}}$ if such prime ideals exist, else we fix $\Delta_{n-d,n-i}^{\mathfrak{A}} = \langle 1 \rangle$; we will also denote $\cap_{0 \leq i \leq k} \Delta_{n-d,n-i}^{\mathfrak{A}}$ by $\Delta_{n-d,\geq n-k}^{\mathfrak{A}}$.

Now, we describe how we define recursively $\Delta_{n-i+1}^{\mathfrak{A}}$ from $\Delta_{n-i}^{\mathfrak{A}}$ for $1 \leq i \leq d$. In the sequel, $\Delta_{n-i,n-j}$ denotes the intersection of prime ideals of co-dimension $n-j$ if such prime ideals exist else we fix $\Delta_{n-i,n-j} = \langle 1 \rangle$.

Our construction works as follows. We consider the algebraic set defined by $\Delta_{n-i,n-i}^{\mathfrak{A}}$ in $\mathfrak{k}^n$ and its equidimensional component of dimension i that we denote by $\mathfrak{V}_{n-i,n-i}$ here after.

We start by constructing the ideal associated to the union of the singular locus of $\mathfrak{V}_{n-i,n-i}$ and the critical locus of Π_i restricted to $\mathfrak{V}_{n-i,n-i}$. If $\Delta^{\mathfrak{A}}_{n-i,n-i} = \langle 1\rangle$ then we let $\mathsf{M}^{\mathfrak{A}}_{n-i} = \langle 1\rangle$ else $\mathsf{M}^{\mathfrak{A}}_{n-i}$ is the ideal generated by the $(n-i)$-minors of $\mathsf{jac}(\mathbf{G}(\Delta^{\mathfrak{A}}_{n-i,n-i}), \mathbf{X}_{\geq i+1})$ and $\Sigma^{\mathfrak{A}}_{n-i+1}$ be the radical ideal $\sqrt{\Delta^{\mathfrak{A}}_{n-i,n-i} + \mathsf{M}^{\mathfrak{A}}_{n-i}}$. By construction, the ideal $\Sigma^{\mathfrak{A}}_{n-i+1}$ is the ideal associated to the union of the singular locus of $\mathfrak{V}_{n-i,n-i}$ and the critical locus of the restriction of Π_i to $\mathfrak{V}_{n-i,n-i}$. Thus, the definition of $\Sigma^{\mathfrak{A}}_{n-i+1}$ does not depend on $\mathbf{G}(\Delta^{\mathfrak{A}}_{n-i+1})$.

Then, we define $\Delta^{\mathfrak{A}}_{n-i+1}$ as $\Sigma^{\mathfrak{A}}_{n-i+1} \cap \Delta^{\mathfrak{A}}_{n-i,\geq n-i+1}$.

As said above, we will consider linear change of variables. Consider a matrix $\mathbf{B}_r = \mathrm{GL}_n(\mathbb{Q})$ of the form $\mathbf{B}_r = \begin{bmatrix} \mathbf{B}' & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{n-r} \end{bmatrix}$, where $\mathbf{B}'$ is square of size r, $\mathbf{I}_{n-r}$ is the identity matrix of size $n-r$. We let $\mathfrak{B} = \mathfrak{A}\mathbf{B}_r$ whose entries are linear forms in the entries of $\mathfrak{A}$; then for $f \in \mathbb{Q}(\mathfrak{A}_{i,j})[\mathbf{X}]$, $\mathsf{Subs}_{\mathfrak{B}}(f)$ denotes the polynomial obtained by substituting in f the entries of $\mathfrak{A}$ by those of $\mathfrak{B}$. If I is an ideal in $\mathbb{Q}(\mathfrak{A}_{i,j})[\mathbf{X}]$, then $I^{\mathbf{B}_r}$ denotes the ideal $\{f(\mathbf{B}_r\mathbf{X}) \mid f \in I\}$ and $\mathsf{Subs}_{\mathfrak{B}}(I)$ denotes the ideal $\{\mathsf{Subs}_{\mathfrak{B}}(f) \mid f \in I\}$.

LEMMA 4. *Let $r \leq i$. If $\Delta^{\mathfrak{A}\mathbf{B}_r}_{n-i} = \mathsf{Subs}_{\mathfrak{B}}(\Delta^{\mathfrak{A}}_{n-i})$, then $\Delta^{\mathfrak{A}\mathbf{B}_r}_{n-i+1} = \mathsf{Subs}_{\mathfrak{B}}(\Delta^{\mathfrak{A}}_{n-i+1})$.*

PROOF. The proof is done by induction. We detail below the induction; the initialization step being obtained by substituting i by $d+1$ in the sequel.

By assumption $\Delta^{\mathfrak{A}\mathbf{B}_r}_{n-i} = \mathsf{Subs}_{\mathfrak{B}}(\Delta^{\mathfrak{A}}_{n-i})$. Recall that these ideals are radical. Consequently, the uniqueness of prime decomposition implies that $\Delta^{\mathfrak{A}\mathbf{B_r}}_{n-i,n-i} = \mathsf{Subs}_{\mathfrak{B}}(\Delta^{\mathfrak{A}}_{n-i,n-i})$ and $\Delta^{\mathfrak{A}\mathbf{B}_r}_{n-i,\geq n-i+1} = \mathsf{Subs}_{\mathfrak{B}}(\Delta^{\mathfrak{A}}_{n-i,\geq n-i+1})$. Thus, to conclude it is sufficient to prove that $\Sigma^{\mathfrak{A}\mathbf{B}_r}_{n-i+1} = \mathsf{Subs}_{\mathfrak{B}}(\Sigma^{\mathfrak{A}}_{n-i+1})$. Let $\mathbf{G} = \mathbf{G}(\Delta^{\mathfrak{A}}_{n-i,n-i})$. Since $\Delta^{\mathfrak{A}\mathbf{B}_r}_{n-i,n-i} = \mathsf{Subs}_{\mathfrak{B}}(\Delta^{\mathfrak{A}}_{n-i,n-i})$, we get $\langle \mathbf{G}^{\mathbf{B}_r}\rangle = \langle \mathsf{Subs}_{\mathfrak{B}}(\mathbf{G})\rangle$. Equality $\langle \mathbf{G}^{\mathbf{B}_r}\rangle = \langle \mathsf{Subs}_{\mathfrak{B}}(\mathbf{G})\rangle$ implies that both ideals define the same algebraic variety $\mathfrak{V}$ in $\mathfrak{k}^n$. By construction, the ideal $\Sigma^{\mathfrak{A}\mathbf{B}_r}_{n-i+1}$ is the ideal associated to the union of the singular locus of $\mathfrak{V}$ and the critical locus of the restriction of Π_i to $\mathfrak{V}$. The same statement occurs for $\mathsf{Subs}_{\mathfrak{B}}(\Sigma^{\mathfrak{A}}_{n-i+1})$ so these ideals coincide. $\square$

Let k be a field; given an ideal $I \subset k[\mathbf{X}]$, we denote by $\mathcal{R}(I)$ the following property: *Let P be a prime ideal appearing in the prime decomposition of $\sqrt{I}$, and r its dimension. Then $k[\mathbf{X}_{\leq r}] \to k[\mathbf{X}]/P$ is integral.*

PROPOSITION 2. *Let $i \in \{1, \ldots, d+1\}$, the ideal $\Delta^{\mathfrak{A}}_{n-i+1}$ satisfies property $\mathcal{R}$, and has dimension at most $i-1$.*

PROOF. We prove the property by decreasing induction on $i = d+1, \ldots, 1$. The case $i = d+1$ is obtained following the Noether Normalization Theorem.

Let us now assume that the property holds for index $i+1$, and prove it for index i. We first establish property $\mathcal{R}(\Delta^{\mathfrak{A}}_{n-i+1})$. The dimension property will follow from it since it implies that Π_i restricted the variety defined by $\Delta^{\mathfrak{A}}_{n-i+1}$ is a finite map. Then, the algebraic Bertini-Sard theorem allows us to conclude.

Preliminaries. Recall that $\Delta^{\mathfrak{A}}_{n-i+1} = \Sigma^{\mathfrak{A}}_{n-i+1} \cap \Delta^{\mathfrak{A}}_{n-i,\geq n-i+1}$. Since $\mathcal{R}(\Delta^{\mathfrak{A}}_{n-i})$ holds by assumption, $\mathcal{R}(\Delta^{\mathfrak{A}}_{n-i,\geq n-i+1})$ holds trivially. Thus, it is sufficient to prove that $\mathcal{R}(\Sigma^{\mathfrak{A}}_{n-i+1})$ holds. Recall also that $\Sigma^{\mathfrak{A}}_{n-i+1}$ is the radical of $\Delta^{\mathfrak{A}}_{n-i,n-i} + \mathsf{M}^{\mathfrak{A}}_{n-i}$ where $\mathsf{M}^{\mathfrak{A}}_{n-i}$ is the ideal generated by the $(n-i)$-minors $M_1, \ldots, M_N$ of $\mathsf{jac}(\mathbf{G}(\Delta^{\mathfrak{A}}_{n-i,n-i}), \mathbf{X}_{\geq i+1})$. We will consider this intersection process incrementally since for proving that $\mathcal{R}(\Delta^{\mathfrak{A}}_{n-i,n-i} + \mathsf{M}^{\mathfrak{A}}_{n-i})$ holds, it is enough to prove that property $\mathcal{R}(\Delta^{\mathfrak{A}}_{n-i,n-i} + \langle M_1, \ldots, M_j\rangle)$ holds for $1 \leq j \leq N$. Note that by assumption $\mathcal{R}(\Delta^{\mathfrak{A}}_{n-i+1})$ holds and we prove below by increasing induction that if $\mathcal{R}(\Delta^{\mathfrak{A}}_{n-i,n-i} + \langle M_1, \ldots, M_j\rangle)$ holds then $\mathcal{R}(\Delta^{\mathfrak{A}}_{n-i,n-i} + \langle M_1, \ldots, M_{j+1}\rangle)$ holds. To simplify notations, we fix $\Delta = \Delta^{\mathfrak{A}}_{n-i,n-i} + \langle M_1, \ldots, M_j\rangle$, $M = M_{j+1}$ and $\Delta' = \Delta + \langle M\rangle$ for $0 \leq j \leq N-1$.

Consider now the prime decomposition $\cap_\ell P_{\ell \leq L}$ of $\sqrt{\Delta}$ for some L and remark that the set of prime components of $\sqrt{\Delta'}$ is the union of the prime components of $\sqrt{P_\ell + \langle M\rangle}$ for $1 \leq \ell \leq L$. Consequently, it is enough to prove that $P_\ell + \langle M\rangle$ satisfies property $\mathcal{R}$ for those ℓ such that $P_\ell + \langle M\rangle \neq \langle 1\rangle$. Thus, as in [26], we partition $\{1, \ldots, L\}$ in four subsets:

- $\ell \in L^+$ if $\dim(P_\ell) = r$ and $M \in P_\ell$;
- $\ell \in L^-$ if $\dim(P_\ell) = r$, $M \notin P_\ell$ and $P_\ell + \langle M\rangle \neq \langle 1\rangle$;
- $\ell \in S$ if $\dim(P_\ell) = r$, $M \notin P_\ell$ and $P_\ell + \langle M\rangle = \langle 1\rangle$;
- $\ell \in R$ if $\dim(P_\ell) \neq r$.

We will prove that $\mathcal{R}(P_\ell + \langle M\rangle)$ holds for $\ell \in L^+ \cup L^-$ while letting $r \leq i$ vary will conclude the proof.

For $\ell \in L^+$, $M \in P_\ell$, $P_\ell + \langle M\rangle = P_\ell$ while $\mathcal{R}(P_\ell)$ holds by assumption; the conclusion follows. Suppose now that $\ell \in L^-$. Since P_ℓ is prime, by Krull's Principal Ideal Theorem, $P_\ell + \langle M\rangle$ has dimension $r-1$ and is equidimensional. By [26, Lemma 1], it is sufficient to prove that the extension $\mathbb{Q}(\mathfrak{A}_{i,j})[\mathbf{X}_{\leq r-1}] \to \mathbb{Q}(\mathfrak{A}_{i,j})[\mathbf{X}_{\leq r-1}]/(P_\ell + \langle M\rangle)$ is integral which is what we do below.

Proving the integral extension. This step of the proof is common with the one of [26, Proposition 1];. we summarize it. By assumption, the extension $\mathbb{Q}(\mathfrak{A}_{i,j})[\mathbf{X}_{\leq r}] \to A_\ell = \mathbb{Q}(\mathfrak{A}_{i,j})[\mathbf{X}_{\leq r}]/P_\ell$ is integral, it is only needed to prove that $P_\ell + \langle M\rangle$ contains a monic polynomial in $\mathbb{Q}(\mathfrak{A}_{i,j})[\mathbf{X}_{\leq r-1}][\![\mathbf{X}_r]\!]$. To this end, the characteristic polynomial of the multiplication by M in A_ℓ is naturally considered and more particularly, we consider its constant term α_ℓ. Since $\ell \in L^-$, M does not divide zero in A_ℓ and α_ℓ is not a constant (and hence it is not zero). Moreover, by Cayley-Hamilton's Theorem, $\alpha_\ell \in P_\ell + \langle M\rangle$. This polynomial α_ℓ is proved to be monic in X_r hereafter.

Consider a matrix $\mathbf{B} = \mathrm{GL}_n(\mathbb{Q})$ which lets invariant the last $n-r$ variables and such that $\alpha_\ell(\mathbf{BX})$ is monic in X_r (recall that $r \leq i$). Following *mutatis mutandis* the reasoning of [26, Sect 2.3] (paragraph entitled *Introduction of a change of variables*), we get that

- the constant term of the multiplication by $M(\mathbf{BX})$ modulo $P_\ell^{\mathbf{B}}$ is $\alpha_\ell(\mathbf{BX})$;
- the constant term of the multiplication by $\mathsf{Subs}_{\mathfrak{B}}(M)$ modulo $\mathsf{Subs}_{\mathfrak{B}}(P_\ell)$ is $\mathsf{Subs}_{\mathfrak{B}}(\alpha_\ell)$;

Note that we have chosen $\mathbf{B}$ such that $\alpha_\ell(\mathbf{BX})$ is monic in X_r. Thus, if we prove that $\alpha_\ell(\mathbf{BX}) = \mathsf{Subs}_{\mathfrak{B}}(\alpha_\ell)$, we are done (recall that $\mathsf{Subs}_{\mathfrak{B}}$ only consists in substituting the entries of $\mathfrak{A}_{i,j}$ with those of $\mathfrak{A}\mathbf{B}$ which do not depend on $X_1, \ldots, X_n$).

Since $\mathbf{B}$ lets invariant the last $n-r$ variables $X_{r+1}, \ldots, X_n$, we get from Lemma 4 that $\Delta^{\mathbf{B}} = \mathsf{Subs}_{\mathfrak{B}}(\Delta)$ and $M^{\mathbf{B}} = \mathsf{Subs}_{\mathfrak{B}}(M)$. The uniqueness of prime decomposition implies that $\{P_\ell^{\mathbf{B}}, \ell \in L\} = \{\mathsf{Subs}_{\mathfrak{B}}(P_\ell), \ell \in L\}$. Moreover, since $\dim(\mathsf{Subs}_{\mathfrak{B}}(P_\ell)) = \dim(P_\ell^{\mathbf{B}}) = \dim(P_\ell)$, we also have

$$\{P_\ell^{\mathbf{B}}, \ell \in L^+ \cup L^- \cup S\} = \{\mathsf{Subs}_{\mathfrak{B}}(P_\ell), \ell \in L^+ \cup L^- \cup S\}$$

The rest of the reasoning is the same as the one of [26]. Indeed, the above equality implies that for $\ell \in L^-$, there exists $\ell' \in L^+ \cup L^- \cup S$ such that $\mathsf{Subs}_{\mathfrak{B}}(P_\ell) = P_\ell^{\mathbf{B}}$. Since $M^{\mathbf{B}} = \mathsf{Subs}_{\mathfrak{B}}(M)$, the characteristic polynomials of $M^{\mathbf{B}}$ modulo $P_{\ell'}^{\mathbf{B}}$ coincides with the characteristic polynomial of $\mathsf{Subs}_{\mathfrak{B}}(M)$ modulo $\mathsf{Subs}_{\mathfrak{B}}(P_\ell)$, so $\mathsf{Subs}_{\mathfrak{B}}(\alpha_\ell) = \alpha_{\ell'}(\mathbf{BX})$. Recall now that α_ℓ is neither 0 nor a constant, then $\ell' \in L^-$. Thus, $\mathsf{Subs}_{\mathfrak{B}}(\alpha_\ell) = \alpha_{\ell'}(\mathbf{BX})$ is monic in X_r as requested.□

As in [26, Subsection 6.4], this property specializes. For $\mathbf{A} \in \mathrm{GL}_n(\mathbb{Q})$, we denote by $\Delta_{n-i+1}^{\mathbf{A}}$ the ideal obtained by substituting the entries of $\mathfrak{A}$ by those of $\mathbf{A}$. The proof of the result below is skipped but follows *mutatis mutandis* the one of [26, Proposition 2].

Proposition 3. *There exists a non-empty Zariski open set $\mathscr{O} \subset \mathrm{GL}_n(\mathbb{C})$ such that for $\mathbf{A}$ in $\mathscr{O}$, the following holds. Let $1 \leq i \leq d+1$, $P^{\mathbf{A}}$ be one of the prime components of $\Delta_{n-i+1}^{\mathbf{A}}$, and r its dimension. Then $\mathbb{C}[\mathbf{X}_{\leq r}] \to \mathbb{C}[\mathbf{X}]/P^{\mathbf{A}}$ is integral.*

Once the above result is proved, one can conclude the proof of properness properties by using a result of [16] relating the properness property and the above normalization result. More precisely, we use [26, Proposition 3] that we restate below in a form that fits with our construction:

Proposition 4. *[26] Let $\mathbf{A}$ be in $\mathrm{GL}_n(\mathbb{C})$ and $1 \leq i \leq d+1$. The following assertions are equivalent.*

- *For every prime component $P^{\mathbf{A}}$ of $\Delta_{n-i+1}^{\mathbf{A}}$, the following holds. Let r be the dimension of $P^{\mathbf{A}}$; then $\mathbb{C}[\mathbf{X}_{\leq r}] \to \mathbb{C}[\mathbf{X}]/P^{\mathbf{A}}$ is integral.*
- *The restriction of Π_r to $V(P^{\mathbf{A}})$ is proper.*

Finally, we define $V_{n-i}^{\mathbf{A}} \subset \mathbb{C}^n$ as the algebraic variety associated to $\Delta_{n-i}^{\mathbf{A}}$ for $0 \leq i \leq d$. For $j \leq i$, we denote by $V_{n-i,n-j}^{\mathbf{A}} \subset \mathbb{C}^n$ (resp. $V_{n-i,\geq n-j}^{\mathbf{A}} \subset \mathbb{C}^n$) the algebraic variety associated to $\Delta_{n-i,n-j}^{\mathbf{A}}$ (resp. $\Delta_{n-i,\geq n-j}^{\mathbf{A}}$). Consider now a connected component C of $V_{n-i}^{\mathbf{A}} \cap \mathbb{R}^n$. It is the union of some connected components $C_1, \ldots, C_k$ of the real algebraic sets $V_{n-i,n-j_1}^{\mathbf{A}} \cap \mathbb{R}^n, \ldots, V_{n-i,n-j_k}^{\mathbf{A}} \cap \mathbb{R}^n$. Consequently, the boundary of $\Pi_i(C)$ is contained in the boundary of $\cup_{1 \leq \ell \leq k} \Pi_i(C_\ell)$. By construction of $V_{n-i+1}^{\mathbf{A}}$, if $j_\ell > i$ then the boundary of $\Pi_i(C_\ell)$ is contained in $\Pi_i(V_{n-i+1}^{\mathbf{A}})$. By construction of $V_{n-i+1}^{\mathbf{A}}$, $V_{n-i+1,n-i+1}^{\mathbf{A}}$ is the union of the singular points of $V_{n-i,n-i}^{\mathbf{A}}$ and the critical locus of Π_i restricted to $V_{n-i,n-i}^{\mathbf{A}}$. Thus, if $j_\ell = i$, the properness of Π_i restricted to $V_{n-i,n-i}^{\mathbf{A}}$ implies that the boundary of $\Pi_i(C_i)$ is contained in the image by Π_i of $C \cap V_{n-i+1,n-i+1}^{\mathbf{A}}$. This leads to the following lemma which concludes the proof.

Lemma 5. *Let $\mathbf{A} \in \mathscr{O} \cap \mathrm{GL}_n(\mathbb{Q})$ be such that for $1 \leq i \leq d+1$ and all prime components $P^{\mathbf{A}}$ of $\Delta_{n-i+1}^{\mathbf{A}}$ the restriction of Π_r to $V(P^{\mathbf{A}})$ is proper and $C^{\mathbf{A}}$ be a connected component of $V_{n-d}^{\mathbf{A}} \cap \mathbb{R}^n$. Then the boundary of $\Pi_i(C^{\mathbf{A}})$ is contained in $\Pi_i(V_{n-i+1}^{\mathbf{A}})$.*

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