On Gröbner Bases over Rings and Residue Class Polynomials Rings with Torsion

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Abstract

Buchberger (1965) introduced Gröbner bases theory for polynomial rings over fields to give an algorithmic technique to determine a vector space basis of the residue class ring of a zero dimensional ideal. This is called as Macauley-Buchberger basis theorem, as Buchberger's result is based on the work by Macaulay (1916). The Macaulay-Buchberger Basis theorem has been extended from polynomial rings over fields to Noetherian rings for free residue class rings in (Francis & Dukkipati, 2014). For this a Gröbner basis characterization has been given that led to an algorithmic method to test whether a residue class polynomial ring over any Noetherian rings is free or not (i.e there exists a module basis or not). This result is also an elegant generalization of the fact that $Z[x]/\langle f\rangle$ is free if and only f is monic, to a mulivariate case, over any Noetherian rings.

In this work (Dukkipati *et al.*, 2014) we generalize the Macaulay-Buchberger basis theorem for residue class ring $A[x_1,\ldots,x_n]/\mathfrak{a}$, in the case when it is finitely generated as an A-module but need not necessarily a free module, where A is a Noetherian ring and $\mathfrak{a}$ is an ideal. This generalization gives us an insight into the nature of generating sets that span A-module $A[x_1,\ldots,x_n]/\mathfrak{a}$ and allows to study the concept of border bases over rings. We present a border division algorithm over rings and prove termination of the algorithm for a special class of border bases called acyclic border bases. We show the existence of such border bases and present some characterizations in this regard.

We believe that this study helps in improving Gröbner basis methods for polynomial rings over rings in the cases, where ideals that give rise to (i) finitely generated residue class polynomial rings that are free, and (ii) finitely generated residue class polynomial rings with torsion.

Keywords
Gröbner bases, Border Bases, Free modules, Ideal lattices

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