Design of regular nonseparable bidimensional wavelets using Gröbner basis techniques*

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Abstract

The design of two-dimensional filter banks yielding orthogonality and linear-phase filters, and generating regular wavelet bases is a difficult task involving algebraic properties of multivariate polynomials. Using cascade forms implies dealing with non-linear optimization. We turn the issue of optimizing the orthogonal linear-phase cascade from [18] into a polynomial problem and solve it using Gröbner basis techniques and computer algebra. This leads to a complete description of maximally flat wavelets among the orthogonal linear-phase family proposed in [18]. We obtain up to 5 degrees of flatness for a 16×16 filter bank, whose Sobolev exponent is 2.11, making this wavelet the most regular orthogonal linear-phase nonseparable wavelet up to the authors' knowledge.

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1 Introduction

A major tool for the representation of 2-D signals such as images is the 4-band perfect reconstruction filter bank. Basically, it consists of four 2-dimensional filters H_0, H_1, H_2, H_3. The input signal is filtered by each filter separately, and each filtered signal is downsampled by two horizontally and vertically, thus generating 4 subsignals, that can be considered as a non-redundant representation of the original signal. This paper addresses the issue of designing these filters. The most common approach consists in designing separable filters, i.e. 2-D filters obtained by tensor product of 1-D filters, since many design techniques are available for 1-D perfect reconstruction filter banks.

In the design, since a perfect reconstruction filter bank implements signal decomposition onto a basis, we want this basis to be orthogonal : it also provides the energy preservation property between space and transform domains. It is also desirable, in image processing, to use linear-phase filters, which means (at least) centro-symmetric or centro-antisymmetric. A major point, at this stage, is that these two properties cannot be simultaneously achieved by separable filter banks, except in the Haar case. Thus, most filter banks used until now are separable and either orthogonal or linear-phase. Since we want the filter bank to yield both properties, we shall consider nonseparable filter banks in this paper. It is already known that nonseparable orthogonal linear-phase filter banks exist [18, 28], but few design examples are available. Actually, the major design techniques for orthogonal nonseparable filter banks (not necessarily linear-phase) are the following :

- Straightforward formulation of the design as an optimization of the filters' coefficients under the quadratic constraints for orthogonality (implying perfect-reconstruction) [36].
- Optimization of cascade forms ensuring orthogonality structurally [18, 29, 28]. It is to be noticed that no complete cascade is until now available in the multidimensional case, due to the lack of a factorization theorem. The main difficulty here appears when trying to optimize the parameters of the cascade.
- State-space matrix representations [32] look to be a very promising approach to the design of orthogonal multidimensional filter banks [33].

The design technique we propose derives from the cascade form approaches. It consists in looking at the optimization issue as solving a set of polynomial equations and in solving these equations using the computer algebra techniques known as Gröbner bases.

It is well known that such filter banks may generate wavelet bases [5, 17]. In this case, the scaling function ϕ is the limit of the subdivision process based on H_0 and satisfies the two-scale equation :

$$\phi(x,y) = \sum_{i,j} \phi(2x-i, 2y-j) H_0(i,j) \tag{1}$$

The wavelets are linear combinations of integer translates of ϕ :

$$\psi_1(x,y) = \sum_{i,j} \phi(2x-i, 2y-j) H_1(i,j) \tag{2}$$

$$\psi_2(x,y) = \sum_{i,j} \phi(2x-i, 2y-j) H_2(i,j) \tag{3}$$

$$\psi_3(x,y) = \sum_{i,j} \phi(2x-i, 2y-j) H_3(i,j) \tag{4}$$

It can be proven [7] that the resulting wavelets can exist and be $N-1$ times continuously differentiable only if the polynomials $H_0, \ldots H_3$ vanish as well as their derivatives up to order N at these aliasing frequencies (of flatness order N. It is known for 1-D dyadic systems that, in practice, imposing these vanishing moments is an efficient way to obtain some regularity [11], so that we expect that designing maximally flat non-separable filter banks will provide regular wavelet bases.

More precisely, we consider a particular family of nonseparable filter banks for sampling matrix $2I$, holding structurally orthogonality and phase-linearity. This family [18] is defined by polynomial matrix products including some angles that can be chosen arbitrarily. The coefficients of H_0, H_1, H_2, H_3, seen as polynomials in variables z_1 and z_2 are thus polynomials in terms of the cosines and sines of the angle parameters. The flatness equations for the polynomials $H_0, \ldots H_3$ are linear combinations of the coefficients of the polynomials $H_0, \ldots H_3$. Thus, the resulting system is polynomial w.r.t. sines and cosines. This shows the principle leading to use techniques for solving polynomial systems in this signal processing context. It should be emphasized that the application of Gröbner basis techniques to filter bank design is very different from [22].

In practice the straightforward application of existing algorithms to the polynomial system obtained by writing the flatness equations in terms of the cascade parameters cannot design filters with support larger than 6×6 and two degrees of flatness. In order to design filter banks with higher regularity, we propose a substitution of variables, splitting the system into two smaller subsystems, which makes possible the design of filters with support up to 16×16 and five degrees of flatness.

The paper is then divided as follows. In section 2, we present the cascade form we use in the sequel, borrowed from [18]. We show how the issue of maximizing the flatness of filters of given size can be turned into solving a polynomial system. Section 3 is devoted to the estimation of the regularity of 2-D wavelets. The algorithm derives from mathematical results [7], and will be used as an analysis tool for the resulting wavelets and filter banks. In section 4, we briefly present computer algebra tools for solving polynomial systems, the Gröbner basis notion, and the algorithms that are currently available. Using these tools, we then present, in section 5, a substitution of variables and a strategy to solve the problem. This provides the minimal size for achieving a given flatness order, the number of remaining free degrees, and a parametrization of the family of the corresponding filters for this flatness order. Examples for different filters' sizes are described in section 6 : they are obtained through an optimization of the remaining free degrees and we describe the resulting filter banks in terms of regularity, frequency characteristics and performances in image compression.

2 Problem statement for maximally flat wavelet among the Kovacevic–Vetterli family

The principle of our approach consists in considering a cascade form ensuring structurally orthogonality and linear-phase and including some degrees of freedom which we optimize so as to maximize the flatness. Straightforward approaches might have been considered, but writing the orthogonality equations in the space domain leads to a system with very high numbers of variables and equations. This cascade form let us reduce the number of variables, and allows to choose them freely, although making any optimization a non-linear problem. We now describe the problem in more detail. This cascade is not complete (i.e : not every orthogonal linear phase filter bank can be written under this form). We first define this family of filter banks. Let us define

$$R_i = \begin{bmatrix} \cos\alpha_i & -\sin\alpha_i & 0 & 0 \\ \sin\alpha_i & \cos\alpha_i & 0 & 0 \\ 0 & 0 & \cos\beta_i & -\sin\beta_i \\ 0 & 0 & \sin\beta_i & \cos\beta_i \end{bmatrix} \qquad W = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} \tag{5}$$

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad D(z_1, z_2) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & z_1 & 0 & 0 \\ 0 & 0 & z_2 & 0 \\ 0 & 0 & 0 & z_1 z_2 \end{bmatrix} \tag{6}$$

$$\mathcal{H}(z) = R_1 W P \prod_{i=2}^{K} D(z_1, z_2) P W R_i W P \tag{7}$$

The filters $H_0 \dots H_3$ are defined as

$$H_i(z) = \mathcal{H}_{i,0}(z_1^2, z_2^2) + \mathcal{H}_{i,1}(z_1^2, z_2^2) z_1 + \mathcal{H}_{i,2}(z_1^2, z_2^2) z_2 + \mathcal{H}_{i,3}(z_1^2, z_2^2) z_1 z_2 \tag{8}$$

where $\mathcal{H}_{i,j}$ denotes the (i, j) component of the matrix $\mathcal{H}$.

It is easy to check that the resulting filters H_0 and H_1 are centrosymmetric while H_2 and H_3 are centro-antisymmetric. Each filter then has linear phase and a square support of size $2K \times 2K$. It can also be checked that the filter bank is orthogonal, meaning that we represent the input signal on an orthogonal basis. Orthogonality implies that the analysis–synthesis system is perfect reconstructing (when there is no quantization of the subband signals). These properties are ensured structurally, whatever K might be. For given K, we have to choose the angles $\alpha_1, \ldots \alpha_K$ and $\beta_1, \ldots, \beta_K$. We aim at having maximally flat filters, while flatness of order N for the filters $H_0 \ldots H_3$ at given points writes as follows : for all $k_1, k_2, k_1 + k_2 \leq N$,

- $\frac{\partial^{k_1+k_2} H_0}{\partial z_1^{k_1} \partial z_2^{k_2}}$ vanishes at $(1,-1)$, $(-1,-1)$, $(-1,1)$;
- $\frac{\partial^{k_1+k_2} H_1}{\partial z_1^{k_1} \partial z_2^{k_2}}$ vanishes at $(1,-1)$, $(1,1)$, $(-1,1)$;
- $\frac{\partial^{k_1+k_2} H_2}{\partial z_1^{k_1} \partial z_2^{k_2}}$ vanishes at $(1,1)$, $(-1,-1)$, $(-1,1)$;
- $\frac{\partial^{k_1+k_2} H_3}{\partial z_1^{k_1} \partial z_2^{k_2}}$ vanishes at $(1,-1)$, $(-1,-1)$, $(1,1)$.

Straightforwardly, the coefficients of filters $H_0, \ldots H_3$ are polynomial w.r.t. the parameters $\cos\alpha_i, \sin\alpha_i, \cos\beta_i, \sin\beta_i$, and the flatness equations are linear w.r.t. the coefficients of the filters, making the whole system of equations, described by K and N, a set of polynomial equations.

3 Regularity estimates for two-dimensional wavelets

We look for wavelets that are continuously differentiable, as many times as possible. However, among the functions that are continuous but not continuously differentiable for instance, some are more regular than others. There are two common definitions of regularity, corresponding to two families of functions' spaces : the Sobolev (resp. Hölder) exponent of a function refers to the index of the smallest Sobolev (Hölder) space it belongs to. Estimating the Sobolev and Hölder exponents of wavelets is not an easy task, but efficient algorithms exist for 1-D wavelets [11, 25].

There are, however, few regularity estimates for nonseparable wavelets in literature : Villemoes proposes in [35] an estimate of Sobolev exponent in the quincunx case, and in [34] an estimate of Hölder exponent in the quincunx case, if this exponent is less than 1. Cohen *et al* and Gröchenig propose in [7] an estimate of Sobolev exponent for general sampling matrices, which we will present more precisely now for the sampling matrix $2I$. A lower bound of the Hölder exponent can also be calculated [20], on the basis of [6].

The estimate needs a preliminary condition : The scaling funtion ϕ and its integer translates must be a Riesz basis of the subspace they span. Assume that the so-called "Cohen's condition"

[5] is met, which is always the case in practice, since we consider low-pass filters that do not vanish in their pass-band. We then define the transfer operator T associated to $|H_0|^2$, which acts on $2\pi\mathbb{Z}^2$-periodic functions as :

$$Tf(\omega) = \sum_{k=0}^{3} \left| H_0\left(\frac{\omega + 2\pi e_k}{2}\right)\right|^2 f\left(\frac{\omega + 2\pi e_k}{2}\right) \tag{9}$$

where

$$\{e_0, e_1, e_2, e_3\} = \{(0,0), (0,1), (1,0), (1,1)\}$$

The finite dimensional space F of trigonometric polynomials whose Fourier coefficients are supported in $[0, 2K-1]^2$ is stable by T. For the sequel T denotes the restriction to F. If the spectral radius of T is 1, if the eigenvalue 1 is the only unitary eigenvalue and has multiplicity 1, then the integer translates of the scaling function are a Riesz basis of the subspace they span. This property is assumed in the sequel.

The Sobolev exponent of the scaling function ϕ is defined as follows, where Φ denotes the Fourier transform of ϕ :

$$s_2 = \sup\left\{ s > 0; \int \left(1 + ||\omega||^2\right)^s |\Phi(\omega)|^2 \, d\omega < \infty \right\} \tag{10}$$

We assume that H_0 has flatness order N, which means that N is the largest integer such that :

$$\forall k \in \{1, \ldots, 3\}, \forall q_1, q_2, \quad q_1 + q_2 \leq N, \frac{\partial^{q_1+q_2} H_0}{\partial \omega_1^{q_1} \partial \omega_2^{q_2}}(\pi e_k) = 0 \tag{11}$$

The Sobolev exponent s_2 of ϕ is given as :

$$s_2 = -\frac{\log \rho}{2 \log 2} \tag{12}$$

where ρ denotes the spectral radius of T restricted on the subspace E of trigonometric polynomials vanishing at $(0,0)$ as well as all their partial derivatives up to order N.

It can be shown [20] that the spectrum of T is made of the spectrum of T restricted to E and of the eigenvalues 2^{-k} for $0 \leq k \leq N$, with multiplicity $k+1$. For example, the Matlab implementation of the Sobolev estimate consists in writing the matrix of T as :

$$A(i,j) = a(2(i/(4K-1)) - (j/(4K-1)), 2(i\%(4K-1)) - (j\%(4K-1))) \tag{13}$$

where $a(k,l)$ denote the Fourier coefficients of $|H_0(\omega)|^2$, / the integer division and % the remainder of the integer division. The spectrum is calculated and ρ is extracted from the set of eigenvalues.

4 Computer Algebra and Gröbner bases

We now review some major algorithms for solving multivariate polynomial systems. The reader is also referred to [16, 30, 8] for a more detailed introduction. In order to give an intuitive presentation of these notions we frequently use analogies with linear algebra well known concepts.

In the following, a polynomial is a finite sum of terms and a term is the product of a coefficient and a monomial.

4.1 Simplification of polynomial systems

Solving linear systems consits in studying vector spaces, and similarly solving polynomial systems consists in studying ideals. More precisely, we define a system of polynomial equations $P_1 = 0, \ldots, P_R = 0$ as a list of multivariate polynomials with rational coefficients in the algebra $\mathbb{Q}[X_1, \ldots, X_N]$. To such a system we associate $\mathcal{I}$, the ideal which is generated by $P_1, \ldots, P_R$; it is the smallest ideal containing these polynomials, and also the set of $\sum_{k=1}^{R} P_k U_k$ where the U_k are in $\mathbb{Q}[X_1, \ldots, X_N]$. Since the P_k vanish exactly at points where all polynomials of $\mathcal{I}$ vanish, it is equivalent to studying the system of equations or the ideal $\mathcal{I}$.

For a set of linear equations, one can compute an equivalent triangular system by "canceling" the leading term of each equation. A similar method can also be done for multivariate polynomials. Of course we have to define the leading term of a polynomial, or in other words *ordering* the monomials. Thus we choose an ordering on monomials compatible with the multiplication. In this paper we only use three kinds of ordering:

- "lexicographic" order : (Lex)

$$x^{\alpha} = x^{(\alpha_1, \ldots, \alpha_N)} <_{Lex} x^{\beta} = x^{(\beta_1, \ldots, \beta_N)} \Leftrightarrow \exists i_0 \forall i = 1 \ldots i_0,\ \alpha_i = \beta_i \text{ and } \alpha_{i_0} < \beta_{i_0} \tag{14}$$

- "degree reverse lexicographic" order : (DRL)

$$x^{(\alpha_1, \ldots, \alpha_N)} <_{DRL} x^{(\beta_1, \ldots, \beta_N)} \Leftrightarrow x^{\left((\sum_k \alpha_k), \beta_N, \ldots, \beta_1\right)} <_{Lex} x^{\left((\sum_k \beta_k), \alpha_N, \ldots, \alpha_1\right)} \tag{15}$$

- "DRL by blocks" order : (DRL,DRL) we split the variables into two blocks, $\alpha = (\alpha_1, \ldots, \alpha_N) = ((\alpha_1, \ldots, \alpha_{N'-1}), (\alpha_{N'}, \ldots, \alpha_N)) = (\alpha', \alpha'')$ for some $N' < N$.

$$x^{(\alpha', \alpha'')} <_{DRL,DRL} x^{(\beta', \beta'')} \Leftrightarrow (x^{\alpha'} <_{DRL} x^{\beta'}) or ((\alpha' = \beta') \ and \ x^{\alpha''} <_{DRL} x^{\beta''}) \tag{16}$$

Now we can define the *leading monomial* (resp. term) of a polynomial as its monomial (resp. term) with highest degree. To cancel the leading terms of the polynomials $p = 9x^2y + \cdots$ and $q = 3xy^2 + \cdots$ we compute $r = yp - 3xq$; r is called the *S-pol* of p and q:

S-pol(polynomial p, polynomial q, $<$ a monomial ordering)

$t_p = \mathrm{LeadMon}(p, <)$

$t_q = \mathrm{LeadMon}(q, <)$

return $\frac{lcm(t_p, t_q)}{t_q} p - \frac{lcm(t_p, t_q)}{t_p} q$

In the special case where the S-pol is simply $r = p - uq$ (e.g. $p = 2x^2y + \cdots$ and $q = xy + \cdots$), we say that p is *reducible* by q and that r is the *reduction* of p by q:

Reducible(polynomial p, polynomial q, $<$ a monomial ordering)

if LeadMon(q,$<$) divides LeadMon(p,$<$) **then return** true

else return false

end if

Reduce(polynomial p, polynomial q, $<$ a monomial ordering)

if Reducible(p,q), **then return** $p - \frac{\mathrm{LeadTerm}(p, <)}{\mathrm{LeadTerm}(q, <)} q$,

else return p

end if

We can extend straightforwardly this definition for reducing a polynomial by a list of polynomials:

Reduce(polynomial p, list of polynomials l=$[q_1, \ldots, q_n$, $<$ a monomial ordering)]

for k **from** 1 **to** n **do**

if Reducible(p,q_k,$<$) **then return** Reduce(Reduce(p,q_k,$<$),l,$<$)

end if

end for

return p.

It should be noted that the output of the reduction of a polynomial by an arbitrary list depends on the order of the polynomials in the list.

4.2 Gröbner bases

We can now give a sketch of the Buchberger [2, 3, 4] algorithm which can be seen as a constructive definition of Gröbner bases:

Gröbner(polynomials $f_1, \ldots f_n$, $<$ a monomial ordering)

$Pairs = \{[f_i, f_j],\ 1 \leq i < j \leq n\}$

while $Pairs \neq \emptyset$ **do**

Choose and remove a pair $[f_i, f_j]$ in $Pairs$

$f_{n+1} = \text{Reduce}(\text{S-pol}(f_i, f_j, <), [f_1, \ldots, f_n], <)$

if $f_{n+1} \neq 0$ **then**

$n = n + 1$

$Pairs = Pairs \bigcup \{[f_i, f_n],\ 1 \leq i < n\}$

end if

end while

return $[f_1, \ldots, f_n]$

Definition 1 *The output G of the algorithm is called a* Gröbner *basis of $\mathcal{I}$ for the order $<$*

Theorem 1 *G has the following properties:*

(i) G is an equivalent set of generators of $\mathcal{I}$.

(ii) A polynomial p belongs to $\mathcal{I}$ if and only if Reduce$(p, G) = 0$

(iii) The output of Reduce(p,G) does not depend on the order of the polynomials in the list. Thus this is a canonical reduced expression modulus $\mathcal{I}$, and the Reduce function can be used as a simplification *function.*

(iv) From G it is easy to compute the number of complex *solutions (counted with multiplicities) of the input system.*

*(v) If $<$ is lexicographic, G has a "*simple form*" (this will be made more precise later).*

Solutions of an algebraic system could be of a variety of kinds that can be classified w.r.t. their algebraic dimension. For example :

- finite number of *isolated points*, in which case we say that the dimension is 0,
- *curves*, the dimension is 1,
- *surfaces*, the dimension is 2.

If a system has different kinds of solutions (e.g isolated points and curves) then the global dimension is the maximum dimension of each component.

Another meaningful interpretation of the dimension is that it corresponds to the remaining free degrees when all of the equations are satisfied.

4.3 Lexicographic Gröbner bases

The computation time depends strongly on the monomial order that is used: In general, Gröbner bases for a lexicographic ordering are much more difficult to compute than the corresponding DRL Gröbner base (we have an intermediate situation for a block (DRL, DRL) ordering). On the other hand, this computational cost is however worth it, because the lexicographic Gröbner basis has a more or less triangular structure which is suitable for further processings. Fortunately, we can compute efficiently lexicographic Gröbner basis with a different method:

First compute a DRL Gröbner basis.

Then change the ordering by applying a special algorithm [13, 14].

4.4 Zero dimensional systems

Note that when $d = 0$ in T the first equation (1) is a univariate polynomial. It is also possible to impose $k_2 = \cdots = k_n = 1$ (Rational Univariate Representation [26]). As a consequence it is possible for zero dimensional system to carry out very efficiently the following :

- count exactly all complex/real roots (with or without multiplicities)[23, 26].
- isolate real roots with the desired precision (no rounding errors).
- compute floating point approximation of complex/real solutions (rounding errors) [1].

Unfortunately the algebriac systems occuring in this paper are not zero dimensional.

4.5 Positive dimensional systems

Since the number of solutions is infinite, things are more difficult; we use the last concepts/algorithms/implement
15]: it is possible to rewrite a lexicographic Gröbner base as a (finite) union of triangular systems [9, 19]:

$$\text{System} = \bigcup_{i=1}^{k} T_i$$

Each T_i is a triangular system of dimension d_i (hence the dimension of the whole system is $\max(d_i)$). The structure of a triangular system of dimension d is (eventually permuting some variables):

$$T \left\{ \begin{array}{ll} x_{d+1}^{k_{d+1}} + \frac{h_1^{(d+1)}(x_1,\ldots,x_d)}{h_0^{(d+1)}(x_1,\ldots,x_d)} x_{d+1}^{k_{d+1}-1} + \frac{h_2^{(d+1)}(x_1,\ldots,x_d)}{h_0^{(d+1)}(x_1,\ldots,x_d)} x_{d+1}^{k_{d+1}-2} + \cdots = 0 & (d+1) \\ x_{d+2}^{k_{d+2}} + \frac{h_1^{(d+2)}(x_1,\ldots,x_{d+1})}{h_0^{(d+2)}(x_1,\ldots,x_{d+1})} x_{d+2}^{k_{d+2}-1} + \frac{h_2^{(d+2)}(x_1,\ldots,x_{d+1})}{h_0^{(d+2)}(x_1,\ldots,x_{d+1})} x_{d+2}^{k_{d+2}-2} + \cdots = 0 & (d+2) \\ \ldots & \\ x_n^{k_n} + \frac{h_1^{(n)}(x_1,\ldots,x_{n-1})}{h_0^{(n)}(x_1,\ldots,x_{d+1})} x_n^{k_n-1} + \frac{h_2^{(n)}(x_1,\ldots,x_{n-1})}{h_0^{(n)}(x_1,\ldots,x_{d+1})} x_n^{k_n-2} + \cdots = 0 & (n) \end{array} \right.$$

where all the $h_i^{(j)}$ are polynomials.

4.6 Computer algebra systems

There is a Gröbner function in every computer algebra system (Maple, Mathematica, Axiom, ...), but it must be emphasized that these implementations are *very* inefficient compared to recent software; even the specialised softwares (Magma, Singular, Macaulay, Asir) are unable to solve the most difficult systems of the paper. Two of the authors have developped efficient C/C++ software in their respective field :

- **Gb** [1] is devoted to the computation of Gröbner basis and triangular systems . To give a rough estimate of the efficiency, one can say that a Gröbner basis computation > 1 sec in Gb is impossible in Maple. For 16×16 filters, we will use FGb[15] the successor.
- **RS**[2] is specialised for the study of real roots. For instance, RS can isolate, by intervals with rational bounds and in few seconds, all of the real roots of an algebraic system (degree 40) whose coefficients contains 300 digits or more.

5 Design of maximally flat filters among the Kovacevic–Vetterli family

Solving the problem of obtaining maximally flat filters divides into many parts :

1. Generating the system of equations.
2. Substituting the variables by using computation of Gröbner basis techniques.
3. Determining, for given K the highest possible, N.
4. Describing the family of the filters satisfying equations (K, N).
5. Choosing the remaining free degrees so as to obtain good properties.
6. Analyzing the resulting filters in terms of time–frequency localization, of regularity and of performances in an image compression system.

[1] http://posso.lip6.fr/GB.html

[2] http://medicis.polytechnique.fr/~rouillie/rs/rs.html

5.1 Straightforward approach

The only example provided in [18] is a solution to the (3,2) system. In fact, this (3,2) problem can be solved by using computer algebra software as a "black box" : let us write the equations as a polynomial system in cosines and sines of the angles. There are 6 angles, so that there are 12 variables in the polynomial system. For each $H_i, \partial H_i/\partial z_1, \partial H_i/\partial z_2$, we consider vanishing moments equations at 3 points, which makes 36 equations altogether, and we have to add the 6 equations imposing $\sin^2 + \cos^2 = 1$. Gröbner basis computation can then be carried out, and a zero-dimensional system is obtained. Triangular systems (in this case the Rational Univariate Representation) provide the roots explicitely, since the first polynomial factorizes in terms of degree four. We thus obtain 64 explicit solutions in the form $[\cos\alpha_1, \ldots, \cos\beta_3, \sin\alpha_1, \ldots, \sin\beta_3]$ e.g. :

$$\left[\frac{\sqrt{2}}{8} + \frac{\sqrt{15}\sqrt{2}}{8}, \frac{7}{8}, \frac{1}{4}, -\frac{\sqrt{2}}{8} + \frac{\sqrt{15}\sqrt{2}}{8}, \frac{\sqrt{15}}{8}, \frac{\sqrt{15}}{4}, -\frac{\sqrt{2}}{2}, -1, -1, \frac{\sqrt{2}}{2}, 0, 0\right].$$

These solutions correspond basically to 2 different filter banks. However, the solution which is different from the one presented in [18] is not better than the previous one in terms of space-frequency localization or of regularity.

Actually, this is the only example where the straightforward approach can be applied, because in other cases the system is not zero-dimensional.

5.2 Generating the system

It takes few lines of code to implement directly in Maple (or a similar system) the matrix formulation described in section 2. But for $K \geq 6$ and $N \geq 4$ it takes several hours to generate the equations and for $K \geq 8$ it is impossible to obtain the results.

The first idea consists in working with 2×2 complex matrices instead of 4×4 real matrices. It is easy to identify a linear transform in $\mathbb{R}^4$ with a linear transform in $\mathbb{C}^2$ if it holds a positive determinant. Not all of the matrices in the product have a positive determinant, making the identification a little more tricky. More precisely the matrix P will be considered simultaneously with a neighbour matrix so as to make the identification possible, as follows :

$$PD(z_1, z_2)P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & z_1 & 0 & 0 \\ 0 & 0 & z_1 z_2 & 0 \\ 0 & 0 & 0 & z_2 \end{bmatrix} \tag{17}$$

and similarly

$$P \begin{bmatrix} 1 \\ z_1 \\ z_2 \\ z_1 z_2 \end{bmatrix} = \begin{bmatrix} 1 \\ z_1 \\ z_1 z_2 \\ z_2 \end{bmatrix} \tag{18}$$

We will then consider, instead of

$$H(z_1^2, z_2^2) \begin{bmatrix} 1 \\ z_1 \\ z_2 \\ z_1 z_2 \end{bmatrix} \tag{19}$$

the following cascade of processings. Using Eq. 18, we start with the vector

$$u_0 = \begin{bmatrix} 1 + iz_1 \\ (z_1 + i)z_2 \end{bmatrix} \tag{20}$$

we consider the matrices one after the other. When considering W, the current vector is multiplied by

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \tag{21}$$

When considering R_l, the current vector is multiplied by

$$\begin{bmatrix} e^{i\alpha_l} & 0 \\ 0 & e^{i\beta_l} \end{bmatrix} \tag{22}$$

When considering $PD(z_1, z_2)P$, the following processing is applied to the current vector $u = (u(1), u(2))'$:

$$u := \begin{bmatrix} \mathcal{R}(u(1)) + iz_1\mathcal{I}(u(1)) \\ z_1 z_2 \mathcal{R}(u(2)) + iz_2\mathcal{I}(u(2)) \end{bmatrix} \tag{23}$$

This approach lets us divide the number of variables by 2 and decrease dramatically the number of terms in each equations (because we replace products of sums by products of monoms). The resulting equations are polynomials in $e^{i\alpha_l}$ and $e^{i\beta_l}$.

We also notice that we consider the derivatives at given points and so it is useless to compute huge polynomials in z_1, z_2, calculate the derivatives and then evaluate them at these points afterwards. A much more efficient way consists in dealing with finite order expansions at these points, directly providing the equations for all derivatives. In addition, as we consider derivatives at points (-1,-1),(1,-1), (1,1) and (-1,1), we observe that each equation writes as a sum of terms such as $e^{\imath(\pm\gamma_1 \pm\gamma_2 \cdots \pm\gamma_K)}$

where γ_i denotes either α_i or β_i; thus each term can be coded efficiently as a 32 bit word for $K \leq 16$ and the multiplication of the two terms becomes a simple bitwise operation. The previous cascade of processings was implemented in C, and the resulting program generates quickly the equations.

5.3 A substitution of Variables

However, there are still a large number of equations: the size of the generated system is more than 60 mega bytes for $K = 8$! We now propose a substitution of variables. Combined with an efficient use of Gröbner bases tools (in particular we use the reduction function as a simplification process), it provides a much nicer formulation of the issue. Basically, we solve the issue by induction for N (detailed below), and we propose the following substitution of variables :

$$\alpha_1 = a_1 - \frac{a_2 - b_2}{2} \tag{24}$$

$$\beta_1 = \frac{a_2 + b_2}{2} \tag{25}$$

$$\alpha_k = \frac{a_k - b_k}{2} - \frac{a_{k+1} - b_{k+1}}{2} \text{ for } k = 2 \ldots K-1 \tag{26}$$

$$\beta_k = \frac{a_{k+1} + b_{k+1}}{2} - \frac{a_k + b_k}{2} \text{ for } k = 2 \ldots K-1 \tag{27}$$

$$\alpha_K = \frac{a_K - b_K}{2} \tag{28}$$

$$\beta_K = b_1 - \frac{a_K + b_K}{2} \tag{29}$$

and we obtain the following results :

Theorem 2 *(i) Equations can be reduced to 2 sub-systems, one depending only on the a_k, and the other one only on b_k. Both sub-systems are identical. In each sub-system there are only K variables (compared to $4K$ in the first formulation).*

(ii) a_1 and b_1 are equal to $-\frac{\pi}{4}$ mod π.

Hence we make a second subtitution of variables :

$$A_1 = B_1 = \frac{1}{2}, \quad A_k = e^{i(a_k - a_1)}, \quad B_k = e^{i(b_k - b_1)} \quad \text{for} \quad k = 2 \ldots K \tag{30}$$

Now, let us briefly describe the induction. For $N = 1$, we observe directly that the system $S(K, 1)$ reduces to the constraint on a_1 and b_1, as already mentioned in [18]. Assume that the reverse lexicographic degree Gröbner basis $\mathrm{GB}_A(K, N-1)$ of the system $S_A(K, N-1)$ is known. It provides directly the reverse lexicographic degree Gröbner basis $\mathrm{GB}(K, N-1)$ of the system $S(K, N-1)$. Using finite order expansions, $S(K, N)$ is obtained as equations in A_k, $1/A_k$, B_k,

$1/B_k$ and i with equation $i^2+1=0$. This system is reduced by the family GB$(K, N-1)$, and further reduced by using the fact that $1/B_k$ is the inverse of B_k. At this point, we observe that the resulting system can be split into two subsystems, $S_A(k,N)$ depending on A_k and $1/A_k$ and $S_B(K,N)$ depending on B_k and $1/B_k$.

By this inductive calculation, we obtain the following as an equivalent system to the system (K,N) :

Theorem 3 *For $K \leq 10$,$N \leq 6$, the system (K,N) that $A_2,\ldots,A_K$ must satisfy is $\{E_2,\ldots,E_N\}$ where:*

$$E_2 \qquad \sum_{k=1}^{K} A_k = 0$$

$$E_3 \qquad \sum_{k=2}^{K} \frac{1}{A_k}\left(\sum_{j=1}^{k-1} A_j - \sum_{j=k+1}^{K} A_j\right) = -\frac{1}{4}$$

$$E_4 \qquad \sum_{k=2}^{K-1} \frac{1}{A_k}\left(\sum_{j=k+1}^{K} A_j\right)\left(\sum_{j=1}^{k-1} A_j\right) = -\frac{1}{8}$$

$$E_5 \qquad \sum_{i=4}^{K}\sum_{j=2}^{i-1} \frac{1}{A_iA_j}\left(\sum_{k=1}^{j-1} A_k - \sum_{i+1}^{K} A_k\right)\left(\sum_{k=j+1}^{i-1} A_k\right) = \frac{4K-7}{32}$$

$$E_6 \qquad \sum_{k=4}^{K} \frac{1}{A_k}\sum_{i=2}^{k-2}\frac{1}{A_i}\left(\sum_{j=i+1}^{k-1} A_j\right)\left(2\left(\sum_{j=1}^{i-1} A_j\right)+4\left(\sum_{j=k+1}^{K} A_j\right)\left(\sum_{j=2}^{i-1} A_j\right)\right) = 3$$

These equations have been checked by computer algebra for $K \leq 10$, and we conjecture that they hold for any K. For $N < 4$ the computation can be done by hand.

This means that all solutions to the problem (K,N) correspond to solutions to the above equations where all coordinates *lie on the unit circle.*

Unfortunately all of the methods described in section 4 give all of the complex solutions of an algebraic system ($|z|^2 = 1$ is not an algebraic equation). So we use the following trick: as the polynomials have rational coefficients, the conjugate of a solution is a solution, hence for a solution with coordinates on the unit circle, the set of the inverses of the coordinates is also a solution. We introduce the new variables A'_k and the equations $A_kA'_k - 1 = 0$; then A'_k must satisfy the same equations as A_k. The resulting system contains all of the admissible solutions we look for (where all coordinates yield modulus 1), and other parasite solutions, but this trick helps in reducing dramatically the number of "bad" solutions.

For a further description of the solutions of this system, we need a list of triangular systems (we got previous results by using only reverse lexicographic degree Gröbner bases) whose computation is much easier. For systems such as (8,5), the computation of the lexicographic Gröbner basis is a very difficult task, making it necessary to use recent and efficient computer algebra algorithms.

Let $\mathcal{S}_{K,N}$ be the system with variables A_k and A'_k (where $A_k A'_k = 1$).
The computation is then carried out in five steps :

1. Computation of the Gröbner basis of $\mathcal{S}_{K,N}$ with DRL order, the variables being ordered as $[A_1, \ldots, A_K, A'_1, \ldots, A'_K]$.
2. Computation of the Gröbner basis of $\mathcal{S}_{K,N}$ with a (DRL,DRL) order and two blocks $[A'_1, \ldots, A'_K]$ and $[A_1, \ldots, A_K]$.
3. In the resulting system just keep the equations involving the A_k only. This a DRL Gröbner basis (elimination theorem).
4. Computation of the lexicographic Gröbner basis of this subsystem "by change of ordering".
5. From the latter basis compute a list of triangular systems

The only restriction is that this algebraic approach describes all of the complex solutions of these systems, while we are only interested in the unitary ones. This point has to be considered for each particular system.

5.4 Maximal flatness for given K and maximally flat filter banks

Statement 1 *For a given $K < 9$, we can compute the maximal N such that the system $\mathcal{S}_{K,N}$ has admissible solutions. The exact values $Nmax$ are reported in Table 1.*

The method is as follows : first we show how to find a candidate for $Nmax$ and then we prove (with computations) that it was correct.

Let us consider the following procedure:

ComputeNmax(integer K)

    $N = 1$
    repeat
        Compute $G_{K,N}$ a DRL basis of $\mathcal{S}_{K,N}$
        if $G_{K,N} = \{1\}$ **or** $G_{K,N}$ is zero-dimensional **then return** $N - 1$
        end if
        $N = N + 1$
    end repeat

If $Nmax$ = ComputeNmax(K), $3 < K < 9$, we claim that $\mathcal{S}_{K,Nmax+1}$ has no admissible solutions.

If $G_{K,Nmax+1} = \{1\}$ then the statement is obvious ; otherwise we introduce the new variable c_K and the equation $2c_K - A_K - \frac{1}{A_K} = 0$, so that $c_K = \cos(A_K)$ is a real number with modulus less than 1 for any unitary A_K ; by calculating a Gröbner basis, a degree-2 equation in c_K is obtained, so that we just have to check by hand that this equation has no real solution with modulus less than 1.

Lemma 1 *(i) $\mathcal{S}_{3,2}$ is zero-dimensional and $\mathcal{S}_{K,Nmax}$ is positive-dimensional for $3 < K < 9$.*

(ii) For $K < 9$, $\mathcal{S}_{K,Nmax}$ has admissible solutions.

When we want to prove that solutions with modulus 1 to the system exist (the system is generally not zero-dimensional) we have to add extra equations to make the system zero-dimensional. We rewrite the system so that we look for real roots, and calculate how many there are. For each system we have found a zero-dimensional system holding modulus 1 roots by adding equations setting some coordinates to 1.

There are many interesting facts reported in Table 1. The first fact is that it is possible to achieve high flatness if the filters' support is large enough. We conjecture that an arbitrary flatness is possible, and we prove it until 5. The second fact is that on the whole the systems are not zero-dimensional. There are an infinite number of maximally flat wavelets, so that we will look for wavelets optimizing a property amongst this set of maximally flat wavelets. This is to be compared to the 1-D dyadic case, where the maximally flat orthogonal wavelets of given size are finitely many [10].

5.5 Optimizing the remaining parameters

In terms of computer algebra, this means that the tools developped for zero-dimensional systems cannot be used directly, and this reminds the applicative need for extending the zero-dimensional tools to arbitrary systems. Therefore, the recent algorithms for the computation of the lexicographic Gröbner basis, and the triangular systems are of the highest interest to us. They provide explicitly all of the parameters as a function of 2 or 4 free degrees. The following theorem reduces the number of systems to be studied :

Theorem 4 *For $2 < K < 9$, all of the admissible solutions of $\mathcal{S}_{K,Nmax}$ are solutions of* **one** *triangular system T_K (see 4.5). The number of free parameters in T_K is equal to the number of free parameters in the whole system $\mathcal{S}_{K,Nmax}$.*

This is a by product of the triangular systems theory and was implemented in Axiom by Marc Moreno Maza [19].

The optimization among the maximally flat wavelets consists in choosing these free degrees according to a criterion. Many possible criteria can be considered, and we have mainly considered 3 of them :

Symmetry : The low-pass filter is structurally centro-symmetric, but we would like to be closer to an axial symmetry for the low-pass filter. This can be turned into a quadratic criterion.

Frequency selectivity : Frequency selectivity : we minimize the energy of the low-pass filter outside $[-a, a] \times [-a, a]$, which means maximizing :

$$\int_{-a}^{a}\int_{-a}^{a} |H_0(\omega_1,\omega_2)|^2 d\omega_1 d\omega_2 = \sum_{i,j,k,l} H_0(i,j)H_0(k,l)\left(\int_{-a}^{a}\cos((i-k)\omega_1)d\omega_1\right)\left(\int_{-a}^{a}\cos((j-l)\omega_2)d\omega_2\right) \tag{31}$$

Energy compaction : We consider the energy compaction for a class of signals :

$$\sum_{i,j}\sum_{k,l} H_0(i,j)H_0(k,l)\mathbb{E}[x(i,j)x(k,l)] \tag{32}$$

with the correlations given by $\mathbb{E}[x(i,j)x(k,l)] = \rho^{|i-k|+|j-l|}$ and $\rho = 0.9$.

These criteria are optimized with respect to remaining free degrees with a conjugate gradient algorithm.

6 Design examples

For each example we give:

- The shape of the solutions i.e. of T_K
- Optimized filters for various criteria and their Sobolev exponent.

6.1 Examples for $N = 2$

After the the substitution of variables, the system (3,2) is very easy to solve, because we just have to find two unitary complex points A_2 and A_3 such that $A_2 + A_3 = -\frac{1}{2}$. There are a finite number of solutions, and they are easy to calculate : we, of course, obtain the filter banks we got by the straightforward application of computer algebra tools looking for real roots of polynomial systems.

If we increase K, we are still able to describe the sets of solutions. For $K = 4$, $N = 2$ is still the maximal possible flatness. There is, for both subsystems, a degree of freedom, which can be seen geometrically. Its optimization for various criteria is easy to carry out. For all criteria mentionned above, the resulting filters hold Sobolev exponents larger than 1, so that the corresponding scaling function and wavelets are at least continuous.

For $K = 5$, we know that the maximally flat wavelets hold $N = 3$. However, we might choose to relax some of the flatness constraints so as to improve some other characteristics. E.g., we study the system (5,2). It can be shown to be of dimension 4, and it has a very nice geometric interpretation, depicted in Fig. 1. It is then clear that for each subsystem two parameters can be chosen, if the angle between A_1 and A_4 is not to close to π. However the optimization of the 4 free degrees is a difficult task, and the solutions to $(5, 3)$ have nice properties in terms of symmetry, frequency localization and energy compaction, so that we could not improve numerically these characteristics by relaxing some of the flatness constraints.

6.2 Examples with $N = 3$

The minimal filter size in order to achieve 3 vanishing moments is 10×10 ($K = 5$). The solutions of the system (5,3) are described by the lexicographic Gröbner basis. In each subsystem, a variable can be freely chosen in $[-s_1, s_1]$ modulo 2π where $s_1 \sim 2.652$. The next variable is obtained by solving a second degree equation, and the other ones by linear equations.

If we optimize the symmetry criterion, the resulting filter is depicted in Fig. 2. The filters obtained by optimising the other criteria are very close to this one, although there are very different filters among the maximally flat ones (see eg in figure 2 the filter obtained by maximizing the energy compaction property). All of these filters hold a Sobolev exponent of around 1.46.

For the system (6,3), it is possible to calculate the lexicographic Gröbner basis and even to write the system as a union of triangular systems, one of them yielding 4 free degrees. However, the filters we obtained by the optimization procedure of the free degrees are similar to the one obtained in the (5,3) case.

6.3 Examples with $N = 4$

The smallest size for achieving 4 vanishing moments is 14×14. Once again, the lexicographic basis provides all variables as a function of 2 free degrees. It consists of an equation in A_6 of degree 4 with coefficients that are polynomials in A_7. The kth equation is implying $A_{7-k}, \ldots A_7$ and is linear in A_{7-k}.

However, the computation of the variables from the free degrees includes solving a degree-4 equation, which means, in general, choosing one of the 4 solutions. This is done numerically, and the problem is that numerical algorithms cannot follow a root, which implies that the variables are not continuous functions of the degrees of freedom. This makes the optimization of the remaining degrees of freedom more difficult, but in any case interesting filter banks can be designed. Different filter banks are obtained, depending on the chosen root of the degree-4 equation. For instance, in figures 4, 5, and 6, we present filters optimizing the symmetry criterion for different roots. They hold similar qualities in terms of symmetry, but are qualitatively different. They hold similar Sobolev exponents, between 1.78 and 1.80.

6.4 Examples with $N = 5$

Calculating the Gröbner basis for the system (8,5) cannot be done without recent algorithms. Despite the numerical solving of a 4th degree equation, very interesting filters have been obtained, such as those depicted in Fig. 9 and Fig. 10. The Sobolev regularity is 2.11. The corresponding scaling function and wavelets are orthogonal, linear-phase and continously differentiable.

6.5 Application to image compression

We now present some experimental results on the application of these filter banks holding simultaneously orthogonality and phase-linearity to image compression. We introduced these new filters in the compression scheme presented in [21]. It consists of scalar quantization of the subband signals followed by a Universal Variable Length Coding [12]. The wavelet packet tree is chosen according to a rate-distortion criterion [24], as well as the quantization steps [27].

We consider `Lena256` at 0.5 bpp. With Daubechies' filters with length from 6 to 16, we obtain a Peak Signal to Noise Ratio between 30.40 and 30.75 dB. With our nonseparable filters, we obtain 28.4 dB with the 6×6 filter bank, around 29.6 dB for the 8×8 filter banks, and between 30.42 and 30.57 dB for the 10×10 to 16×16 filter banks. Different filter banks hold very close performances, so that changing the image or the coding scheme may change the hierarchy. In other words all these filters may be considered as equivalent. This is illustrated by figure 8, depicting the rate-distortion curves for separable Daubechies filters of length 14, nonseparable filter with support 6×6 from [18] and nonseparable filter bank from figure 6.

Among the non-separable filter banks, the best results are obtained with the 10×10 filter bank which has been optimized w.r.t. energy compaction (Fig. 3). The 14×14 filter bank in Fig. 6 follows, and then the other 10×10 and 14×14 filter banks, and then the 16×16 filter bank. This means that 16×16 does not ensure enough resolution in space domain.

These results do not take into account the subjective quality of the images. Subjectively, the images encoded by the non-separable filter bank schemes cannot be said to be better or worse than the ones from good separable schemes.

7 Conclusion

The results of our study concern four scientific communities :

- From a computer algebra applications point of view, it is one of the major applications of computer algebra in digital signal processing, and this area looks to be a promising application field for computer algebra, especially the filter bank area. In addition, the problem has been solved using the most recent techniques for the Gröbner bases computation.
- From the point of view of the wavelet field, it is the first example of bidimensional orthogonal linear-phase wavelets which are continuously differentiable.
- From the point of view of the filter bank design, the design examples are convincing : the resulting filters have good frequency characteristics, and it has been shown that the remaining free degrees can be chosen so as to optimize various criteria.
- From the image compression point of view, it is interesting to have new filter banks holding simultaneously orthogonality, centrosymmetry and regularity, and in addition with good frequency characteristics. Relating our results to these of [31] suggests that improved compression efficiency can be achieved by future work on design techniques for such non separable filters.

Further work might also include the study of other cascade forms for filter banks, or a proof that arbitrarily high flatness and regularity can be achieved in this filter bank family.

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References

[1] Auzinger and Stetter. An elimination algorithm for the computation of all zeros of a system of multivariate polynomial equations. *Int. Series of Numerical Math.*, 86:11–30, 1998.

[2] B. Buchberger. *Ein Algorithmus zum Auffinden der Basiselemente des Restklassenringes nach einem nulldimensionalen Polynomideal.* PhD thesis, Innsbruck, 1965.

[3] B. Buchberger. An Algorithmical Criterion for the Solvability of Algebraic Systems. *Aequationes Mathematicae*, 4(3):374–383, 1970. (German).

[4] B. Buchberger. *"Groebner bases : an algorithmic method in polynomial ideal theory"*. Recent trends in multidimensional systems theory. Reider ed. Bose, 1985.

[5] A. Cohen and I. Daubechies. Non-separable bidimensional wavelet bases. *Revista Mat. Iberoamericana*, 9(1):51–137, 1993.

[6] A. Cohen and I. Daubechies. "A new technique to estimate the regularity of refinable functions". *pre-print*, 1995.

[7] A. Cohen, K. Gröchenig, and L. Villemoes. "Regularity of multi-variate refinable functions". *pre-print*, 1995.

[8] D. Cox, J. Little, and D. O'Shea. *Ideals, varieties, and algorithms : an introduction to computational algebraic geometry and commuataive algebra.* Undergraduate texts in mathematics. Springer Verlag, 1992.

[9] D. Lazard. Systems of algebraic equations (algorithms and complexity). In D. Eisenbud and L. Robbiano, editor, *Computational Algebraic Geometry and Commutative Algebra, Proceedings of Cortona Conference.* Cambridge University Press, October 1993.

[10] I. Daubechies. "Orthogonal bases of compactly supported wavelets". *Communications on Pure and Applied Mathematics*, XLI:909–996, 1988.

[11] I. Daubechies. *"Ten Lectures on Wavelets"*. Number 61 in CBMS-NSF Series in Applied Mathematics. SIAM, Philadelphia, 1992.

[12] P. Delogne and B. Macq. "Universal variable length coding for an integrated approach to image coding". *Annales des télécommunications*, 46(7-8), 1991.

[13] J. C. Faugère. *"Résolution des systèmes d'équations polynomiales"*. PhD thesis, Université Paris VI - Pierre et Marie Curie, 1994.

[14] J. C. Faugère, P. Gianni, D. Lazard, and T. Mora. "Efficient computation of zero-dimensional Groebner basis by change of ordering". *Journal of Symbolic Computation*, 1994.

[15] Faugère J.C. A new efficient algorithm for computing Gröbner Basis (F4). Technical report, Task 3.3.2.1 Frisco report, 1997. preprint.

[16] J. Davenport and Y. Siret and E. Tournier. *Calcul Formel.* Masson, 1993. 2^e édition révisée.

[17] J. Kovacevic and M. Vetterli. "Nonseparable multidimensional perfect reconstruction filter banks and wavelet bases for $\mathbb{R}^n$". *IEEE Transactions on Information Theory*, 38(2):533–555, March 1992.

[18] J. Kovacevic and M. Vetterli. "Nonseparable two- and three-dimensional wavelets". *IEEE Transactions on Signal Processing*, 43(5):1269–1272, May 1995.

[19] M. Moreno Maza. *Calculs de Pgcd au-dessus des Tours d'Extensions Simples et Résolution des Systèmes d'équations Algébriques.* PhD thesis, Université de Paris 6, 1997.

[20] F. Moreau de Saint-Martin. *"Bancs de filtres et ondelettes : synthèses, adaptativité et applications".* PhD thesis, CEREMADE, Université Paris IX-Dauphine, 1997.

[21] P. Onno. *"Bancs de filtres et quantification vectorielle sur réseau : Étude conjointe pour la compression d'images".* PhD thesis, Université de Rennes I, France, 1996.

[22] H. Park, T. Kalker, and M. Vetterli. Groebner bases and multidimensional fir multirate systems. *Journal of Multidimensional Systems and signal*, 1996.

[23] P. Pedersen, M-F. Roy, and A. Szpirglas. "Counting real zeros in the multivariate case". *Computational Algebraic Geometry, F.Essette, A. Galigo (editors)*, .(.):203–223, 1993.

[24] K. Ramchandran and M. Vetterli. "Best wavelet packet bases in a rate-distortion sense". *IEEE Transactions on Image Processing*, 2(2):160–175, April 1993.

[25] O. Rioul. "Simple regularity criteria for subdivision schemes". *SIAM Journal Math. Anal.*, 23(6):1544–1576, November 1992.

[26] F. Rouillier. *"Algorithmes efficaces pour l'étude des zéros réels des systèmes polynomiaux".* PhD thesis, Université de Rennes I, May 1996.

[27] Y. Shoham and A. Gersho. "Efficient bit allocation for an arbitrairy set of quantizers". *IEEE Transactions on Acousticcs, Speech and Signal Processing*, 36, September 1988.

[28] D. Stanhill and Y. Y. Zeevi. "Two-dimensional linear-phase orthogonal filter-banks and wavelets". *pre-print*, 1995.

[29] D. Stanhill and Y. Y. Zeevi. "Two-dimensional orthogonal wavelets with vanishing moments". *pre-print*, 1995.

[30] T. Becker and V. Weispfenning. *Groebner Bases, a Computationnal Approach to Commutative Algebra.* Graduate Texts in Mathematics. Springer-Verlag, 1993.

[31] S. Venkataranam and B. C. Levy. "nonseparable orthogonal linear phase perfect reconstruction filter banks and their application to image compression". *Proc. of Int. Conf. on Image Processing*, pages 334–338, 1994.

[32] S. Venkataranam and B. C. Levy. "State space representations of 2-D FIR lossless matrices". *IEEE Transactions on Circuits and Systems–II: Analog and DIgital Signal Processing*, 41(2):117–131, February 1994.

[33] S. Venkataranam and B. C. Levy. "A comparison of design methods for 2-D FIR orthogonal perfect reconstruction filter banks". *IEEE Transactions on Circuits and Systems–II: Analog and Digital Signal Processing*, 42(8):525–536, August 1995.

[34] L. F. Villemoes. "Continuity of nonseparable quincunx wavelets". *Appl. Comput. Harmon. Anal.*, To appear.

[35] L. F. Villemoes. "Sobolev regularity of wavelets and stability of iterated filter banks". In Yves Meyer and Sylvie Roques, editors, *Progress in wavelet analysis and applications*, pages 243–251. Editions frontières. Proceedings of Int. Conf. "Wavelets and Applications", Toulouse, France, 1992.

[36] E. Viscito and J. P. Allebach. "The analysis and design of multidimensional FIR perfect reconstruction filter banks for arbitrary sampling lattices". *IEEE Transactions on Circuits and Systems*, 38(1):29–41, January 1991.

K	3	4	5	6	7	8
maximal N	2	2	3	3	4	5
dimension	0	2	2	4	2	2

Table 1: Maximal flatness N for a filter bank with size $2K$ from [18] and dimension of the corresponding system.

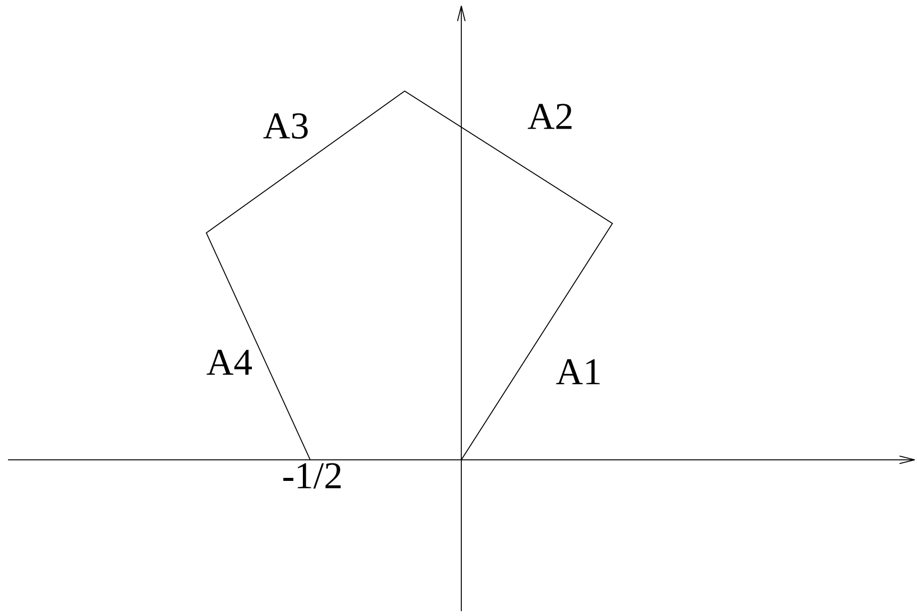


Figure 1: Geometric interpretation for solving the system (5,2).

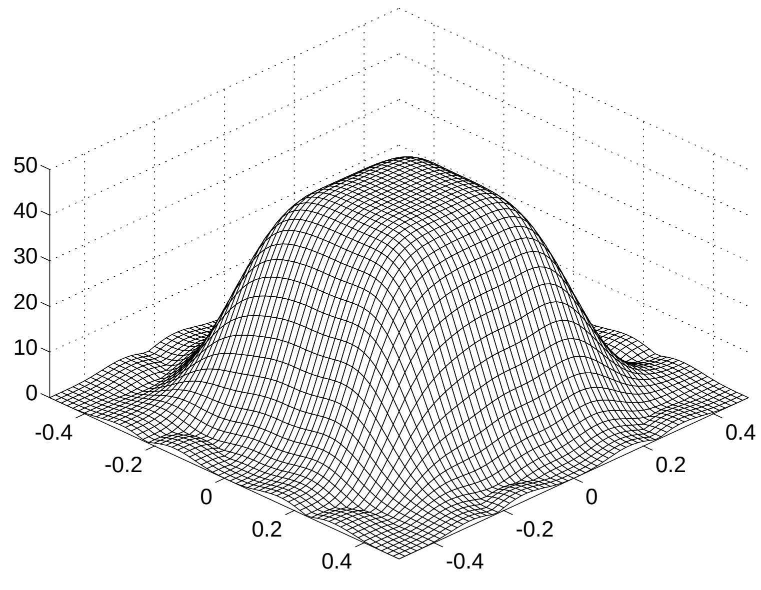


Figure 2: Frequency response of a 10×10 filter with flatness order 3, optimized w.r.t. symmetry criterion.

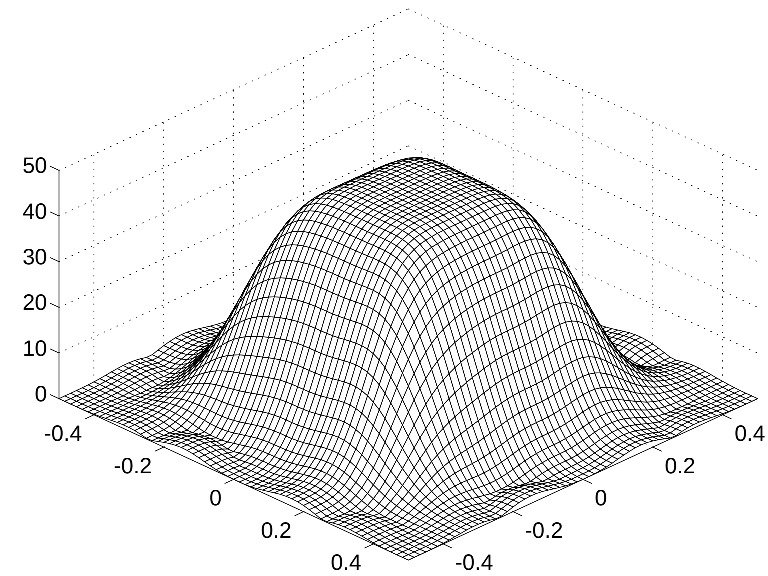


Figure 3: Frequency response of a 10×10 filter with flatness order 3, optimized w.r.t. the energy compaction property.

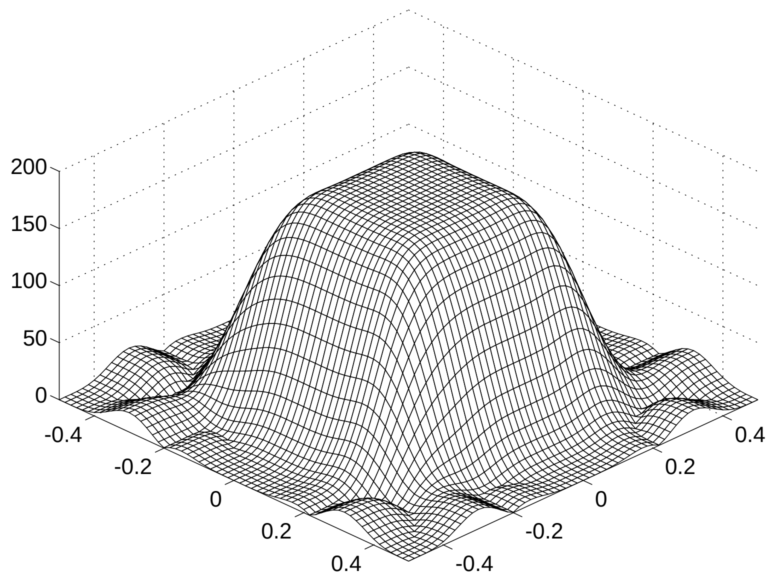


Figure 4: Frequency response of a 14×14 filter with flatness order 4, optimized w.r.t. symmetry criterion (first root in maple sense).

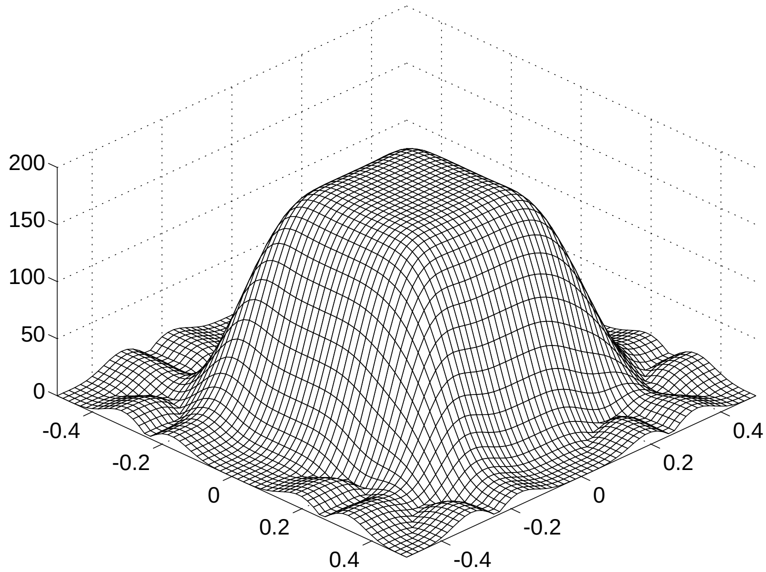


Figure 5: Frequency response of a 14×14 filter with flatness order 4, optimized w.r.t. symmetry criterion (second root in maple sense).

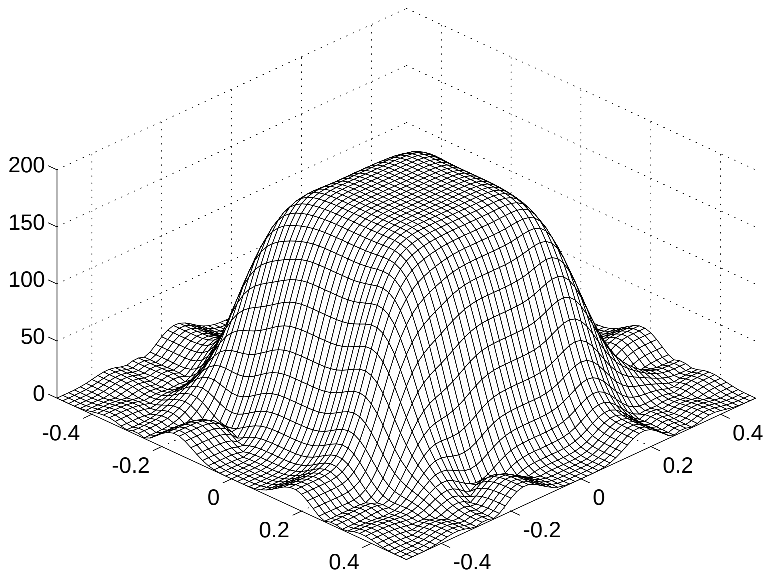


Figure 6: Frequency response of a 14×14 filter with flatness order 4, optimized w.r.t. symmetry criterion (third root in maple sense).

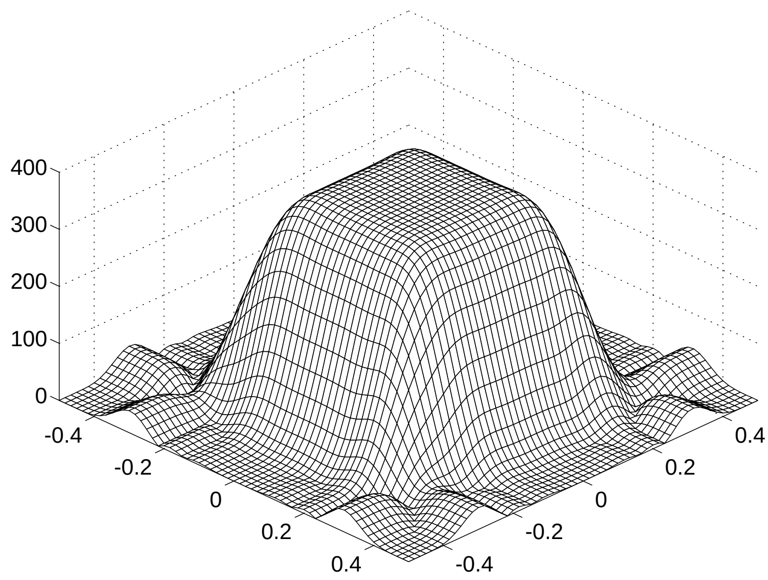


Figure 7: Frequency response of a 16×16 filter with flatness order 5, optimized w.r.t. symmetry criterion.

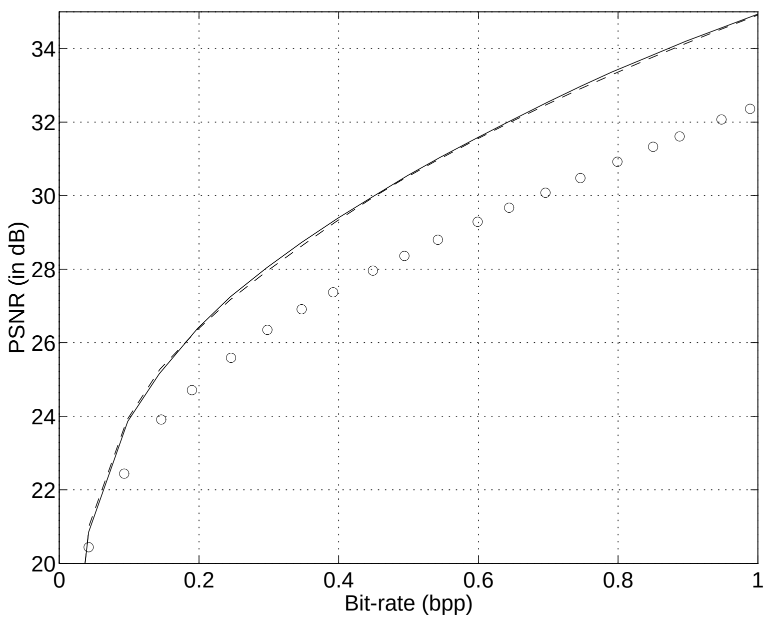
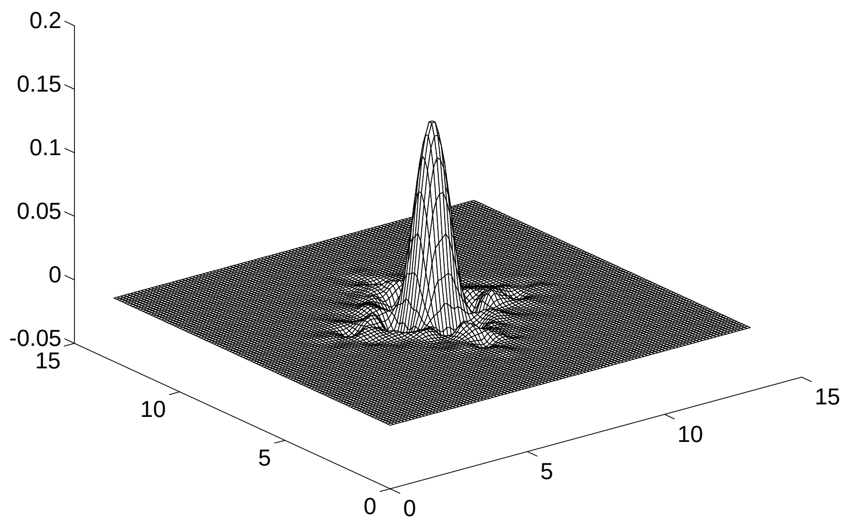


Figure 8: Rate-distortion curves for separable Daubechies filters of length 14 (dashed line), nonseparable filter with support 6×6 from [18] (circles) and nonseparable filter bank from figure 6 (solid line).



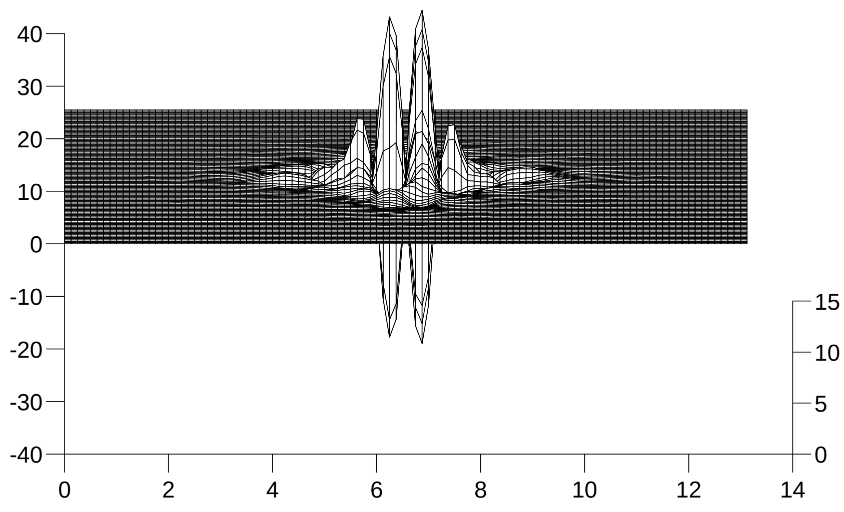


Figure 9: Scaling function and wavelets (H_0 and H_1) associated to the 16×16 filter bank depicted in Fig. 7. It is a solution to the (8,5) problem, whose Sobolev exponent is 2.11.

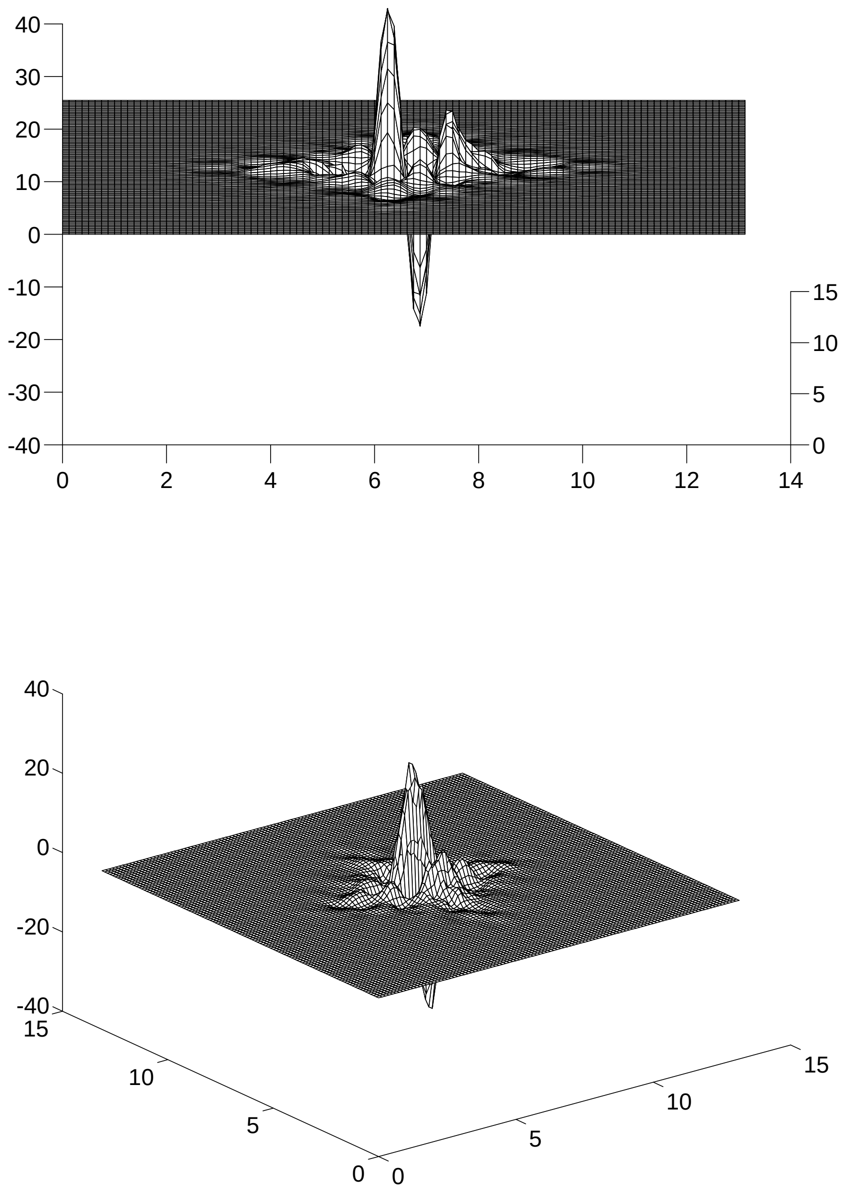


Figure 10: Scaling function and wavelets (H_2 and H_3) associated to the 16×16 filter bank depicted in Fig. 7. It is a solution to the (8,5) problem, whose Sobolev exponent is 2.11.