Lecture 2-13-1 - Polynomial systems, computer algebra and applications

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Solving Algebraic Systems with Symmetries

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Symmetries

System invariant by the action of an Abelian group

Definition

Let I be an ideal. I is said to be *stable* under the action of G (G-*stable*) if: $\forall f \in I, \forall A \in G \quad f^A \in I$

Action of $GL_n(k)$ on polynomials.
G is a finite subgroup of $GL_n(k)$.
Let X be the column vector whose entries are $x_1, \ldots, x_n$.

For f a polynomial and $A \in G$, let f^A be the polynomial obtained by substituting the components of $A.X$ to $x_1, \ldots, x_n$.
Since $(f^A)^B = f^{AB}$, we obtain an action of G on the ring of polynomials

Remark

The action of G preserves the homogeneous components.

Main focus: G is an Abelian Group

System invariant by the action of an Abelian group

Consider the following system: 5 degree 3 equations in 5 variables: invariant by the action of $G = C_5$ (ground field is $\mathbb{F}_{65521}$):

$f_1 =$
$y_1^3 + y_2^3 + y_3^3 + y_4^3 + y_5^3 + 52524\, y_1\, y_5{}^2 + 52524\, y_1{}^2 y_2 + 52524\, y_2{}^2 y_3 + 52524\, y_3{}^2 y_4 + 52524\, y_4{}^2 y_5 +$
$19910\, y_2{}^2 y_4 + 19910\, y_1{}^2 y_3 + 37058\, y_1{}^2 y_4 + 30323\, y_1{}^2 y_5 + 30323\, y_1\, y_2{}^2 + 12774\, y_1\, y_2\, y_3 +$
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$f_2, f_3, f_4, f_5 =$ same shape $\cdots$

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$f_2, f_3, f_4, f_5 =$ same shape $\cdots$

Not **Generic** at all!
The system has 125 solutions.
How to use the symmetry ?

Abelian Group G

Theorem

Any finite commutative group G is uniquely isomorphic to a product $\mathbb{Z}/q_1\mathbb{Z} \times \cdots \times \mathbb{Z}/q_\ell\mathbb{Z}$ with $q_1|\ldots|q_\ell$.

Definition

Following the notations of the previous theorem, the integer $e = q_\ell$ is called the exponent of the group and is the lowest common multiple of the orders of the elements of the group.

When $\ell = 1$ and $n = q_1$ so that G is the n cyclic group.

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Theorem

Let G be a cyclic group of order n. Let ω be a primitive e-th root of 1. The subgroup G is diagonalizable, meaning that there exists a matrix P in $GL_n(K)$, such that the group $P^{-1} G P = \{P^{-1} A P \mid A \in G\}$ is a diagonal group.

Example: cyclic Group G

Let C_n be the subgroup of $\mathfrak{S}_n$ generated by the n-cycle $\sigma = (1\ 2\ \ldots\ n)$. C_n is a cyclic group of order n, embedded in $GL_n(k)$ and generated by:

$$M_\sigma = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ 1 & 0 & 0 & \ldots & 0 \end{pmatrix}$$

The cyclic group C_n is diagonalizable:
Then if we denote $K = k(\omega)$ where ω is a primitive n-root of 1, with the base-change matrix $P = (\omega^{ij})_{i,j \in \{1,\ldots,n\}}$.
The matrix associated to the cycle $(1 \ldots n)$ becomes the diagonal matrix $D_\sigma = \mathrm{diag}(\omega, \ldots, \omega^{n-1}, 1)$.

Grading induced by a diagonal matrix group

$G = \mathbb{Z}/q_1\mathbb{Z} \times \cdots \times \mathbb{Z}/q_\ell\mathbb{Z}$ with $q_1 | \ldots | q_\ell$.

Proposition

For every monomial m and for each i, there exists a unique $\mu_i \in \{0, \ldots, q_i - 1\}$ such that $m^{D_i} = \omega^{\frac{e}{q_i}\mu_i} m$.

☞ We take μ_i in $\mathbb{Z}/q_i\mathbb{Z}$

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Definition

The k-tuple $(\mu_1, \ldots, \mu_k) \in \prod \mathbb{Z}/q_i\mathbb{Z}$ is said to be the G-degree of m and is denoted $G\!-\!degree(m)$.

Cyclic Group G

C_3 is the matrix group generated by the diagonal matrix $D_\sigma = \mathrm{Diag}(\omega, \omega^2, 1)$ where ω is a primitive third root of 1.
Each x_i has G-degree i mod 3, so

$$G\text{-degree}\left(\prod x_j^{\alpha_j}\right) = \sum j\,\alpha_j \bmod 3$$

Hence, $x_1x_2x_3$ (resp. $x_1x_2^2$) has G-degree 0 (resp. 2).

The repartition into same G-degree is as follows :

G-degree	0	1	2
monomials	$1, x_3, x_3^2, x_1x_2$ $x_3^3, x_1x_2x_3, x_2^3, x_1^3$	x_1, x_1x_3, x_2^2 $x_1x_3^2, x_2^2x_3, x_1^2x_2$	x_2, x_2x_3, x_1^2 $x_2x_3^2, x_1^2x_3, x_1x_2^2$

Solving systems invariant by the action of an Abelian group

We diagonalize the group:

$$G = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \text{ compute } Q \text{ s.t. } Q\,G\,Q^{-1} = \begin{bmatrix} w^2 & 0 & 0 & 0 & 0 \\ 0 & w^4 & 0 & 0 & 0 \\ 0 & 0 & w & 0 & 0 \\ 0 & 0 & 0 & w^3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

where $w^5 = 1$. ☞ New variables $Q \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix} = \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$

Solving systems invariant by the action of an Abelian group

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$$G = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \text{ compute } Q \text{ s.t. } Q\,G\,Q^{-1} = \begin{bmatrix} w^2 & 0 & 0 & 0 & 0 \\ 0 & w^4 & 0 & 0 & 0 \\ 0 & 0 & w & 0 & 0 \\ 0 & 0 & 0 & w^3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

where $w^5 = 1$. ☞ New variables $Q \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix} = \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$

$g_1 = 41\,x_0^3 + 9\,x_0\,x_1\,x_4 + 7\,x_0\,x_2\,x_3 - 17\,x_1^2x_3 + 28\,x_1\,x_2^2 + 15\,x_2\,x_4^2 + 44\,x_3^2x_4 - 21\,x_0^2 - 42\,x_1\,x_4 - 27\,x_2\,x_3 + 22\,x_0 - 4120$

$g_2, g_3, g_4, g_5 =$ same shape $\cdots$

- The new system is sparse: length$(g_1) = 12 \ll 56 =$ length(f_1)
- Support of the polys: monomials $x_i x_j x_k$ s.t. $i + j + k = 0 \mod n$
  $\longrightarrow$ New grading: G-degree$(x_{i_1} \cdots x_{i_k}) = i_1 + \cdots + i_k \mod n$
  $g(w^0 x_0, w^1 x_1, \ldots, w^{n-1} x_{n-1}) = w^{G-degree(g)} g(x_0, x_1, \ldots, x_{n-1})$
  $\longrightarrow$ Here all the polynomials of G-degree 0

G-homogeneity

For algorithms: the S-polynomial of two G-homogeneous polynomials is also G-homogeneous

Definition

A polynomial f is said to be G-homogeneous if all monomials of f share the same G-degree $(\mu_1, \ldots, \mu_k)$. In this case, we set G-degree$(f) = G$-degree$(\mathrm{LM}(f)) = (\mu_1, \ldots, \mu_k)$.

G-*homogeneity*

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Proposition

If f is G-homogeneous and m is a monomial, then $m\,f$ is also G-homogeneous. Moreover, $G\text{-degree}(m\,f) = G\text{-degree}(m) + G\text{-degree}(f)$.

G-homogeneity

The cornerstone of the new Abelian-F5 algorithm is that the S-polynomial of two G-homogeneous polynomials is G-homogeneous:

Theorem

Let f, g be two G-homogeneous polynomials. The S-polynomial of f and g is also G-homogeneous of G-degree: G-degree$(LM(f) \vee LM(g))$. Where $LM(f) \vee LM(g) =$ lowest common multiple of $LM(f)$ and $LM(g)$.

Test in a CAS

We consider the cyclic group C_n:

Home Work

- write the matrix M_G of G
- Compute P such that

$$P^{-1}\ M_G\ P = D \text{ a diagonal matrix}$$

- write a function to change the variables
- Apply the change of variables to some interesting polynomial, for instance:

$$x_1 + x_2 + \cdots + x_n$$

Test 1

We will use the well known *Cyclic-n problem*. The ideal I generated by:

$$(I)\begin{cases} f_1 = x_1 + \cdots + x_n \\ f_2 = x_1x_2 + x_2x_3 + \cdots + x_nx_1 \\ \vdots \\ f_{n-1} = x_1x_2\ldots x_{n-1} + x_2\ldots x_nx_1 + \cdots + x_nx_1\ldots x_{n-2} \\ f_n = x_1x_2\ldots x_{n-1}x_n - 1 \end{cases}$$

The ideal I is invariant under the cyclic group C_n, since each h_i satisfies $h_i^{M_\sigma} = h_i$

Home Work

- write the equations
- change the variables
- compute the G-degree of each equations
- Are the polynomials G-homogeneous ?

Test 2:Random Systems

We consider a system of $f_1, \ldots, f_n$ equations in $\mathbb{F}_p[x_1, \ldots, , x_n]$ which are invariant by the action of cyclic group C_n

Home Work

- Generate the equations using the operator:

$$R(f) = \frac{1}{|G|} \sum_{\sigma \in G}^{\sigma} f$$

- change the variables
- compute the G-degree of each equations

Test 3: NTRU (basic problem of several PQC cryptosystem)

p is a prime number
$f_1 = \sum_{i=0}^{n-1} a_i x^i$ with $a_i \in \{0, 1\}$
$f_2 = \sum_{i=0}^{n-1} b_i x^i$ with $b_i \in \{0, 1\}$

Then $Pub = f_1 \times (f_2)^{-1} \mod (x^n - 1) \mod p$

Goal: find a polynomial $f = \sum_{i=0}^{n-1} x_i x^i$ with $x_i \in \{0, 1\}$ such that:

all the coefficients of $Pub \times f \mod (x^n - 1) \mod p$ are in $\{0, 1\}$

Home Work

- write the original algebraic equations
- change the variables
- compute the G-degree of each equations
- Are the polynomials G-homogeneous ?

Fundamental Theorem

G is a diagonal group, and I is a G-stable ideal generated by $f_1, \ldots, f_m$. A Grbner basis computation preserves the G-degree, but the polynomials f_i are not necessarily G-homogeneous. Our aim here is to prove that the G-homogeneous components of the f_i are in I, and so to compute a Grbner basis of I, we take the G-homogeneous components of generators of I as inputs.

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Definition

Let I be an ideal. I is said to be *stable* under the action of G (G*-stable*) if: $\forall f \in I, \forall A \in G \quad f^A \in I$

Definition

An ideal J is said to be G-homogeneous if for any polynomial $f \in J$, its G-homogeneous components are also in J.

Fundamental Theorem

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Theorem

An ideal is G-homogeneous if and only if it is G-stable.

Remark

True also when $G = \{1\}$

Test 1,2,3

If $G = C_n$ then

$$f = \sum_{i=0}^{n-1} f^{(i)} \qquad \text{where} \quad G - \text{degree}(f^{(i)}) = i$$

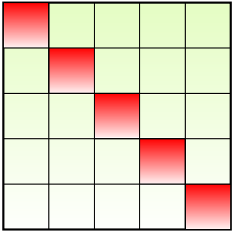
Home Work

- split the equations into G-homogeneous components:

Speedup the computation

Abelian Group $\approx$ Multi-homogeneous :

Use the new Grading to split the matrices



Instead of **one** matrix in degree d

$$\mathcal{M}_d$$

we can split $\mathcal{M}_d$ wrt G-degree $0, 1, 2, 3, 4$.

Theorem ([F., Svartz 2013])

$\mathbf{I} = (f_1, \ldots, f_m)$ a 0-dimensional ideal, invariant under an Abelian Group G. Divides the GB complexity by: $|G|^3$

☞ *Provide dedicated F_5 and FGLM algorithms.*

Abelian F5

Abelian-F_5 (homogeneous-case)

Input: The set $\hat{G}$ of $G_{\mathscr{D}}$-degrees, homogeneous and $G_{\mathscr{D}}$-homogeneous polynomials $(f_1,\ldots,f_m)$ with degrees $d_1 \leq \ldots \leq d_m$ and a maximal degree D.
Output: the elements of degree at most D of a Gröbner basis of $(f_1,\ldots,f_i)$ for $i = 1,\ldots,m$.

---

for i **from** 1 **to** m **do** $\mathscr{G}_i := \emptyset$ **end for**
for d **from** d_1 **to** D **do**
  for g **in** $\hat{G}$ **do**
    $M_{d,0,g} := \emptyset$, $\tilde{M}_{d,0,g} := \emptyset$
      for i **from** 1 **to** m **do**
        case
          $d < d_i)$ $M_{d,i,g} := \tilde{M}_{d,i-1,g}$
          $d = d_i)$ **if** $g = \deg_{G_{\mathscr{D}}}(f_i)$ **then**
              $M_{d,i,g} :=$ add new row f_i to $\tilde{M}_{d,i-1,g}$ with index $(i,1)$
                else
              $M_{d,i,g} := \tilde{M}_{d,i-1,g}$
            end if
          $d > d_i)$ $M_{d,i,g} :=$add new row $m.f_i$ for all monomials m of degree $d - d_i$ with $\deg_{G_{\mathscr{D}}}(m) = g - \deg_{G_{\mathscr{D}}}(f_i)$ that do not appear as leading monomials in the matrix $\tilde{M}_{d-d_i,i-1,u-\deg_{G_{\mathscr{D}}}(f_i)}$ to $\tilde{M}_{d,i-1,g}$ with index (i,m).
        end case
        Compute $\tilde{M}_{d,i,g}$ by Gaussian elimination from $M_{d,i,g}$.
        Add to $\mathscr{G}_i$ all rows of $\tilde{M}_{d,i,g}$ not reducible by $\mathrm{LM}(\mathscr{G}_i)$.
      end for
  end for
end for
return $\mathscr{G}_1,\ldots,\mathscr{G}_m$

Faster?

Consider the following system: 5 degree 3 equations in 5 variables: invariant by the action of $G = C_5$ (ground field is $\mathbb{F}_{65521}$):

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$f_2, f_3, f_4, f_5 =$ same shape $\cdots$

The system has 125 solutions.

Solving Systems with Symmetries

Recall that we want to solve the following system: 5 degree 3 equations in 5 variables which are invariant by the action C_5

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Diagonalize the group !
Change of variables

$g_1 = 41\,x_0^3+9\,x_0\,x_1\,x_4+7\,x_0\,x_2\,x_3-17\,x_1^2x_3+28\,x_1\,x_2^2+15\,x_2\,x_4^2+44\,x_3^2x_4-21\,x_0^2-42\,x_1\,x_4-27\,x_2\,x_3+22\,x_0-4120$

$g_2, g_3, g_4, g_5 =$ same shape $\cdots$

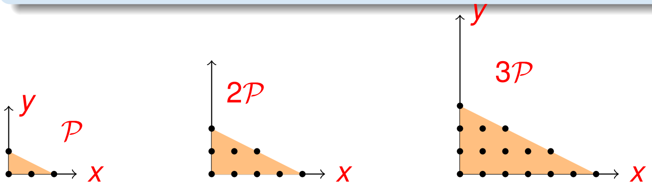
☞ Use the sparsity !

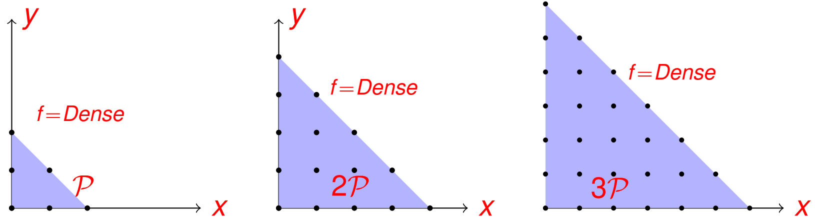
New Unified Approach : Sparse Gröbner basis

with PJ Spaenlehauer and J Svartz - 2014

Unified approach based on monomial sparsity

- Consider only monomials in the initial Support: polytope $\mathcal{P}$
- Multiply these monomials $\rightsquigarrow 2\mathcal{P} = \{u \times v \mid (u, v) \in \mathcal{P}^2\}$





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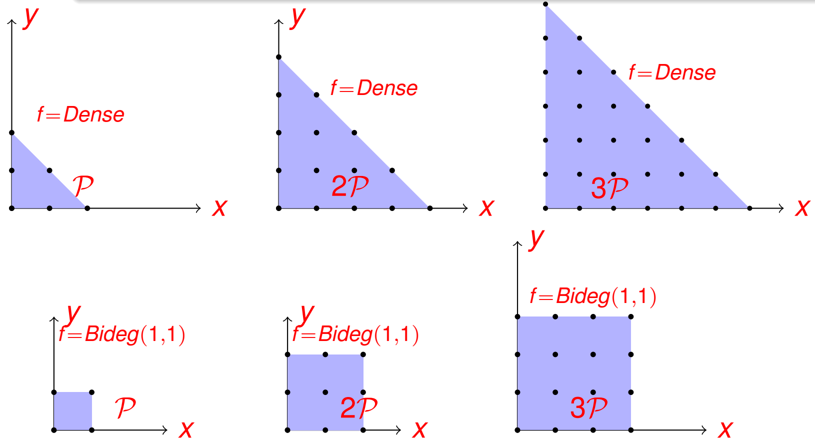
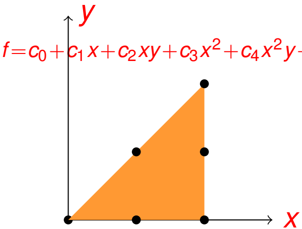


New Approach ! Sparse Polynomials

with PJ Spaenlehauer and J Svartz



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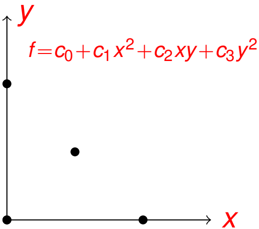
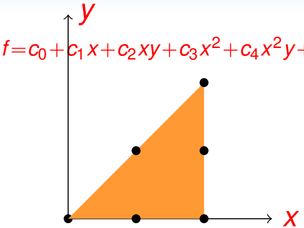


We want to keep the initial structure!

New Approach ! Sparse Polynomials



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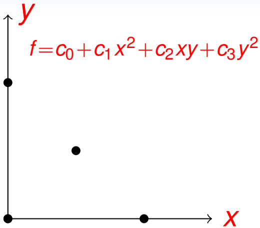
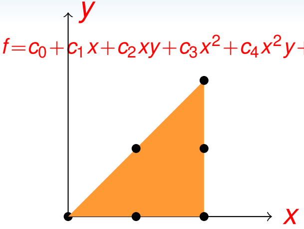


We want to keep the initial structure!

- Monomials of degree 1:
  $\mathcal{M}_1 = \text{Support}(f)$
- Monomials of degree 2:
  $\mathcal{M}_2 = \{u \times v \mid (u, v) \in \mathcal{M}_1 \times \mathcal{M}_1\}$
- ...
- Monomials of degree d:
  $\mathcal{M}_d = \{u \times v \mid (u, v) \in \mathcal{M}_{d-1} \times \mathcal{M}_1\}$

New Approach ! Sparse Polynomials



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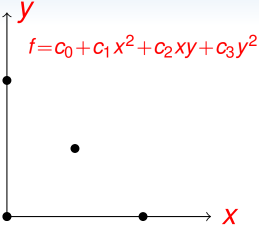

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- ...
- Monomials of degree d:
  $\mathcal{M}_d = \{u \times v \mid (u, v) \in \mathcal{M}_{d-1} \times \mathcal{M}_1\}$

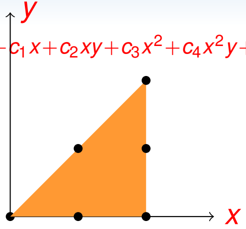
Macaulay Matrix in degree d

$$\begin{array}{cc} & m_1 > m_2 > \cdots > m_k \\ M_d = \begin{array}{c} t_{1,1} f_1 \\ t_{1,2} f_1 \\ \vdots \\ t_{2,1} f_2 \\ \vdots \end{array} & \left(\begin{array}{c} \cdots \\ \cdots \\ \operatorname{coeff}(t\, f_i, m_j) \\ \cdots \\ \vdots \end{array} \right) \end{array}$$

all products $t\, f_i$, $t \in \mathcal{M}_{d-\deg(f_i)}$

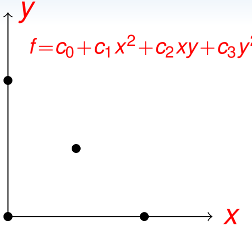
New Approach ! Sparse Polynomials

$f = c_0 + c_1 x + c_2 xy + c_3 x^2 + c_4 x^2 y + c_5 x^2 y^2$



or

$f = c_0 + c_1 x^2 + c_2 xy + c_3 y^2$



We want to keep the initial structure!

- dedicated matrix-F_5 algorithm

Goal

Under algebraic assumptions: m eqs with the same support

- complexity ?
- Hibert Series?

Macaulay Matrix in degree d

$$M_d = \begin{array}{c} \\ t_{1,1} f_1 \\ t_{1,2} f_1 \\ \vdots \\ t_{2,1} f_2 \\ \vdots \end{array} \begin{array}{c} m_1 > m_2 > \cdots > m_k \\ \left(\begin{array}{c} \cdots \\ \cdots \\ \mathrm{coeff}\,(t\, f_i, m_j) \\ \cdots \\ \vdots \end{array} \right) \end{array}$$

all products $t\, f_i$, $t \in \mathcal{M}_{d-\deg(f_i)}$

Solving with symmetries using sparsity

Initial support $\mathcal{P} = \{h_1, \ldots, h_{12}\} =$ Support$(g_i) = \{x_i x_j x_k$ s.t. $i + j + k = 0 \mod 5\} \longrightarrow \#\mathcal{P} = 12$

We have to estimate $d_{\max}$?

- Monomials of degree 1: $\#\mathcal{P} = 12$
- Monomials of degree 2:
  $2\mathcal{P} = \{u \times v \mid (u, v) \in \mathcal{P} \times \mathcal{P}\} \longrightarrow \#2\mathcal{P} = 68$
- ...
- Monomials of degree d: $d\mathcal{P} = \{u \times v \mid (u, v) \in (d-1)\mathcal{P} \times \mathcal{P}\}$

Compute the Hilbert series of the monomial ring:

$$H_R(z) = 1 + \sum_{d>0} \#(d\mathcal{P})\, z^d = \frac{z^4 + 6\,z^3 + 11\,z^2 + 6\,z + 1}{(1-z)^6}$$

Compute the Hilbert series of the monomial ring:

$$H_R(z) = 1 + \sum_{d>0} \#\mathcal{M}_d\, z^d = \frac{z^4+6\,z^3+11\,z^2+6\,z+1}{(1-z)^6}$$
$$= 1 + 12\,z + 68\,z^2 + 254\,z^3 + 730\,z^4 + 1756\,z^5 + \cdots$$

Since we have 5 equations of "degree" 1, the Hilbert series is

$$H(z) = H_R(z)(1-z)^5$$
$$= 1 + 7\,z + 18\,z^2 + 24\,z^3 + 25\,z^4 + 25\,z^5 + 25\,z^6 + \cdots$$

Hence we have only $25 = \frac{125}{|G|}$ solutions
and the maximal degree $d_{\mathrm{max}} = 4$.

☞ We can run the sparse matrix F_5 and compute the minimal polynomial of M_t (where $t = x_0^2$) of degree 25:
$t^{25} + 62732\,t^{24} + 26240\,t^{23} + 63778\,t^{22} + 38558\,t^{21} + 9283\,{h_8}^{20} + 29068\,t^{19} + 49606\,t^{18} + 34528\,t^{17} + 22383\,t^{16} + 11568\,{h_8}^{15} + 8861\,t^{14} + 38583\,t^{13} + 60089\,t^{12} + 23443\,t^{11} + 62330\,{h_8}^{10} + 38047\,t^{9} + 41549\,t^{8} + 42497\,t^{7} + 32676\,t^{6} + 13919\,t^{5} + 22256\,t^{4} + 25537\,t^{3} + 61988\,t^{2} + 108\,t + 60264$ then recover the values of $m \in \mathcal{P}$, and the values of $x_0, x_1, \ldots, x_4$.

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