Lecture 2-13-1 - Polynomial systems, computer algebra and applications

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Regular Sequence
Algebraic Cryptanalysis of HFE (first part)
Link betwen Critical Pairs and Linear Algebra
Characterizations of Gröbner Bases
F_4
F_5

2022 - 2023 – MPRI

HFE Problem

Matsumoto, T., Imai, H.
"Public quadratic polynomial-tuples for efficient signature-verification and message-encryption".
EUROCRYPT '88.

J. Patarin.
"Hidden Fields Equations (HFE) and Isomorphisms of Polynomials (IP): Two New Families of Asymmetric Algorithms".
EUROCRYPT '96.

Secrets:

- two $n \times n$ invertible matrices S and T with coefficients in $\mathbb{F}_2$
- sparse univariate polynomial of degree D

$$H(X) = \sum_{s \in \mathcal{S}} c_s X^s$$

where $c_s \in \mathbb{F}_{2^n}$
and $\mathcal{S} = \{2^i \mid 0 \leqslant i \text{ and } i \leqslant D\} \cup \{2^i + 2^j \mid 0 \leqslant i < j \text{ and } i + j \leqslant D\}$
Particular case: $\mathcal{S} = \{2^i + 2^j\}$ then $H(X)$ is a monomial.

Finite Field Extension

Definition (Finite Field Extension)

The field $\mathbb{F}_{p^n}$ where p prime and $n \geqslant 1$ may be explicitly constructed in the following way. One first chooses an irreducible polynomial P in $F_p[X]$ of degree n (such an irreducible polynomial always exists). Then the quotient ring $\mathbb{F}_{p^n} = \mathbb{F}_p[X]/(P)$ of the polynomial ring $\mathbb{F}_p[X]$ by the ideal generated by P is a field of order p^n.

In our case: $p = 2$, P is of degree n any $x \in \mathbb{F}_{2^n}$ can be written $\sum_{i=0}^{n-1} x_i \omega^i$ where $x_i \in \mathbb{F}_2$ (where ω is the class of X in $\mathbb{F}_p[X]/(P)$).

Example

$\mathbb{F}_4 = \mathbb{F}_2[X]/(X^2 + X + 1)$.

HFE Problem: Public Key

We define the polynomial ring $R = \mathbb{F}_2[X_1, \ldots, X_n]$ and we substitute X by $\sum_{i=0}^{n-1} X_i \omega^i$ into the HFE polynomial $H(X)$:

$$H(\sum_{i=0}^{n-1} X_i \omega^i) = \sum_{i=0}^{n-1} H_i(X_1, \ldots, X_n) \omega^i$$

where each $H_i \in R$ is a polynomial of degree 2.
The next step is to mix the variables:

$$\begin{pmatrix} X_1 \\ \cdots \\ X_n \end{pmatrix} = T \times \begin{pmatrix} x_1 \\ \cdots \\ x_n \end{pmatrix}$$

and the equations:

$$\begin{pmatrix} P_1 \\ \cdots \\ P_n \end{pmatrix} = S \times \begin{pmatrix} H_1 \\ \cdots \\ H_n \end{pmatrix}$$

HFE Public Key

$(P_1, \ldots, P_n) \in R^n$ is the Public Key.

HFE Problem: Public Key

$$\begin{pmatrix} X_1 \\ \cdots \\ X_n \end{pmatrix} = T \times \begin{pmatrix} x_1 \\ \cdots \\ x_n \end{pmatrix}$$

and the equations:

$$\begin{pmatrix} P_1 \\ \cdots \\ P_n \end{pmatrix} = S \times \begin{pmatrix} H_1 \\ \cdots \\ H_n \end{pmatrix}$$

In other words, we compose S, H and T:

$$S(H(T(X))) = (P_1(x_1, \ldots, x_n), \cdots, P_n(x_1, \ldots, x_n))$$

HFE Encrypt/Verify

Encrypt

Assume that $M = (M_1, \ldots, M_n) \in \mathbb{F}_2^n$ is then message. Then

$$\begin{pmatrix} C_1 \\ \cdots \\ C_n \end{pmatrix} = \begin{pmatrix} P_1(M_1, \ldots, M_n) \\ \cdots \\ P_n(M_1, \ldots, M_n) \end{pmatrix}$$

is the ciphertext.

This operation is very fast since it is just the evaluation of quadratic polynomials.

Exercice

This operation can be done with linear algebra. Why ?

HFE Decrypt/Sign

Decrypt

Assume that $C = (C_1, \ldots, C_n) \in \mathbb{F}_2^n$ is then ciphertext.
To decrypt the ciphertext C, we first find all solutions Z to the univariate equation

$$H(Z) = S^{-1}C,$$

next we compute $T^{-1}Z$ to recover the original message.

Hence we have to solve a degree D univariate equation.
Find the roots of a polynomial of degree F with coefficients in $\mathbb{F}_{2^n}$ can be done (see [5] for instance) in $\mathcal{O}(\mathbf{M}(d)\log(d))$ operations in $\mathbb{F}_{2^n}$ where $\mathbf{M}(d)$ is the cost of polynomial multiplication.
In practice, we cannot take arbitrarily big value for the degree of the univariate polynomial (say $d \leqslant 1024$).

HFE Challenge

For instance: $n = 80$, $d = 96$ for the first HFE Challenge.

Algebraic Attack

The direct attack is obvious: given $(M_1, \ldots, M_n)$ we have to solve the following polynomials system:

$$\begin{cases} P_1(x_1, \ldots, x_n) = M_1 \\ \qquad\qquad \ldots \\ P_n(x_1, \ldots, x_n) = M_n \end{cases}$$

Algebraic Attack

$(M_1, \ldots, M_n)$ is given.
More precisely since the solutions are to be found in $\mathbb{F}_2$ we have to solve the following system:

$$\begin{cases} P_1(x_1, \ldots, x_n) = M_1 \\ \qquad \ldots \\ P_n(x_1, \ldots, x_n) = M_n \\ x_1^2 - x_1 \\ \quad \ldots \\ x_n^2 - x_n \end{cases}$$

Question

What is the complexity of solving the algebraic system?
What is the complexity of computing the Gröbner basis.

Simple Estimation of the Complexity

From a complexity point of view the two important parameters are: d the maximal degree occurring in the computation and the size N_d of the Macaulay matrix in degree d.
Then the whole complexity is simply N_d^ω where $2 \leqslant \omega \leqslant 3$ is the cost of linear algebra.
Since we add the field equations $x_i^2 - x_i$ it means that all the monomials are squarefree, so that

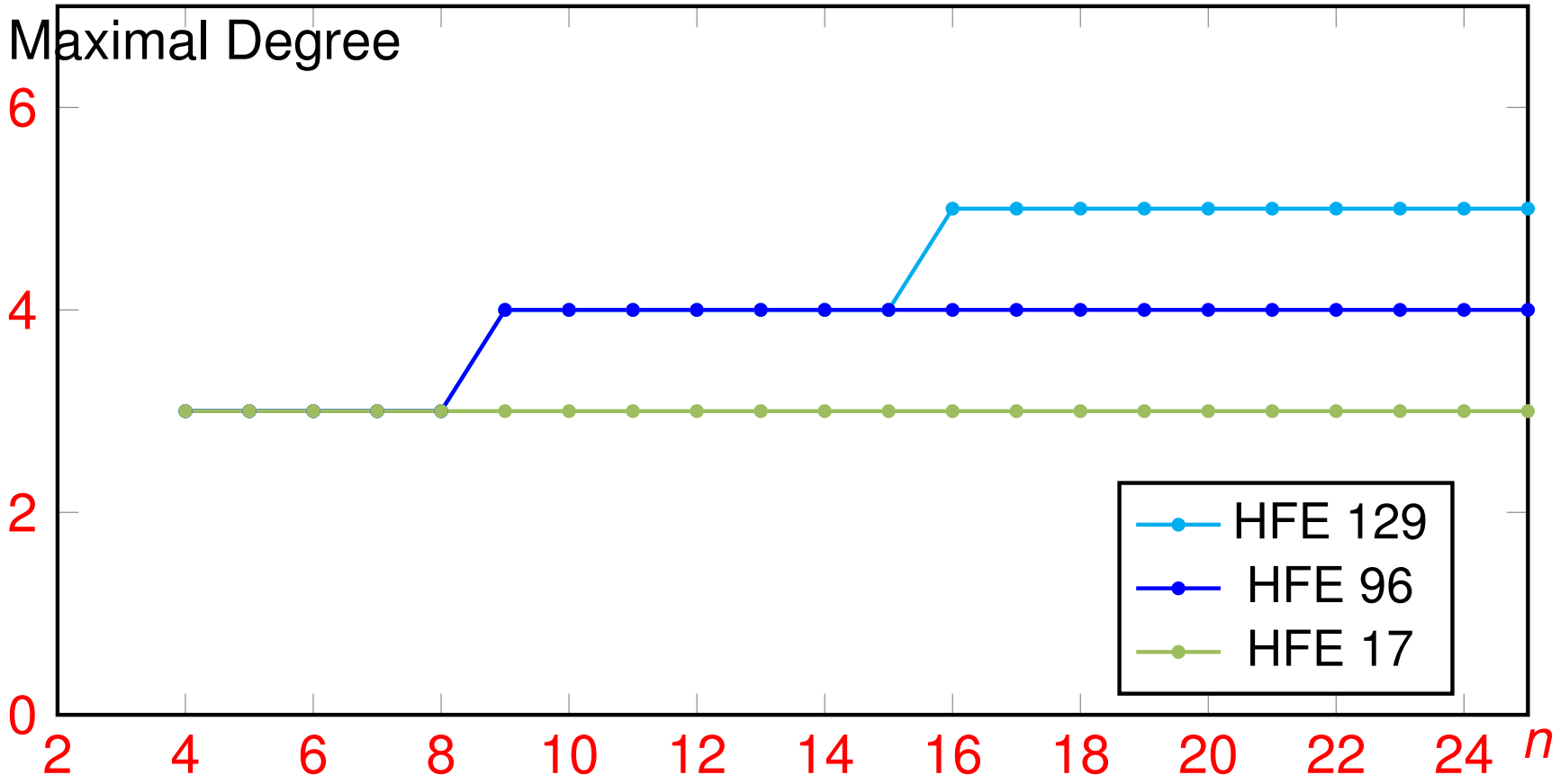
$$N_d = \binom{n}{d}$$

Hence, the main goal is to estimate d.
We begin by running some experiments.

Experimental Complexity

D	16	17	33	96	128	129	257	384	512	513
Max degree	3	4	4	4	4	5	5	5	5	6

Relation between *D* and the maximal degree.

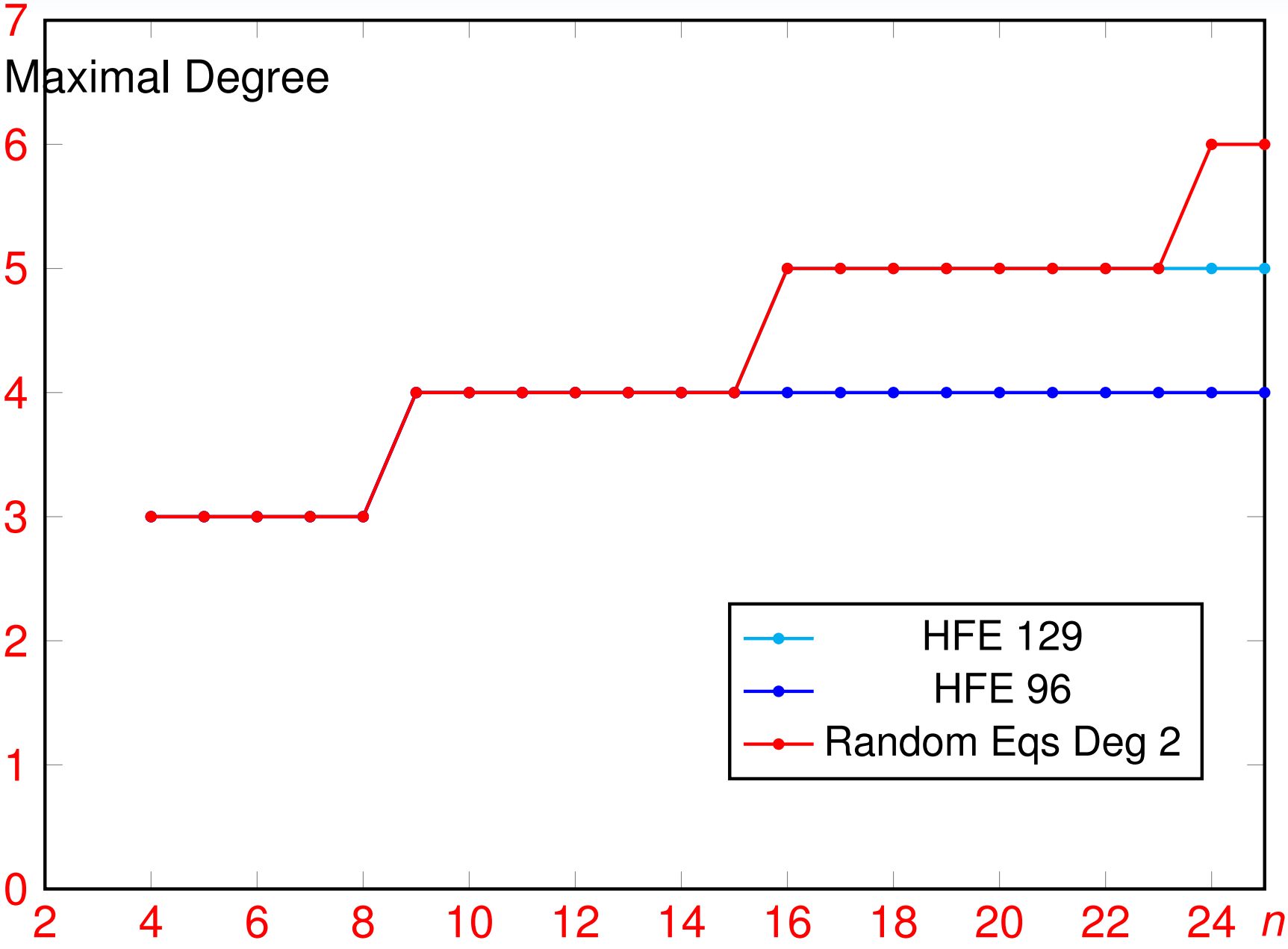


Experimental Complexity

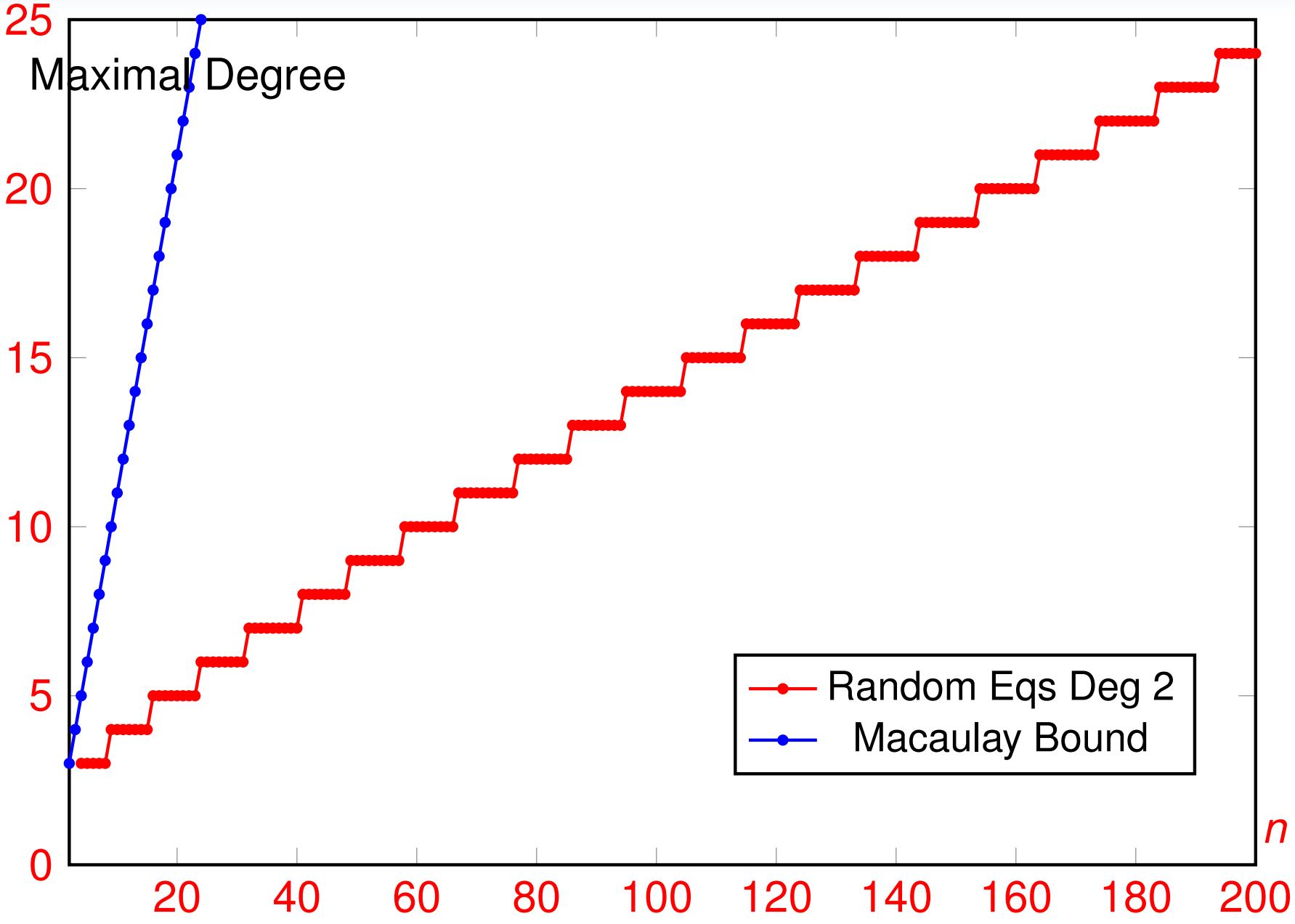
n	14	15	16	17	23	24	25
degree	4	4	5	5	5	6	6
nb of rows	1695	1379	8840	11424	40480	223124	278875

Maximal degree occurring in Gröbner for *random* systems.

Experimental Complexity



Complexity


25
20
15
10
5
0
Maximal Degree
Random Eqs Deg 2
Macaulay Bound
20
40
60
80
100
120
140
160
180
200
n

F_4

F_4

Matrix representation of polynomials

Definition

If $F = [f_1, \ldots, f_m]$ is a vector of m polynomials and $<$ an admissible ordering, $\mathrm{T}_<(F) = [t_1, \ldots, t_l]$ the monomials in the support of F sorted for $<$. The matrix representation of $M_{T_<(F)}(F)$ wrt F is:

$$M(F) = \begin{array}{c|ccc} & t_1 & t_2 & t_3 \\ f_1 & \cdots & \cdots & \cdots \\ f_2 & \cdots & \cdots & \cdots \\ f_3 & \cdots & \cdots & \cdots \end{array}$$

$$M(F)_{f_i,t_j} = \mathrm{coeff}\left(f_i, t_j\right)$$

Moreover, $M(F)$ satisfies the following equation:

$$F = M(F) \cdot T_<(F)$$

Polynomial representation of a matrix

Definition

If M is a matrix of size $l \times m$ with coefficients in $\mathbb{K}$ and $X = [t_1, \ldots, t_m]$ is a vector of terms, then the polynomial representation of M wrt X is the vector of l polynomials given by:

$$F = M \cdot X$$

Example (Cyclic 4 Problem)

The monomial ordering is DRL

$$F = \left[\begin{array}{c} f_1 = abcd - 1, f_2 = abc + abd + acd + bcd, \\ f_3 = ab + bc + ad + cd, f_4 = a + b + c + d \end{array} \right]$$

The matrix representation of $F_1 = [f_3, bf_4, df_4]$ is:

$$A_1 = M(F_1) = \begin{array}{c|ccccccc} & ab & b^2 & bc & ad & bd & cd & d^2 \\ df_4 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ f_3 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ bf_4 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \end{array}$$

Example (Cyclic 4 Problem)

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Macaulay matrix

Definition (Macaulay matrix [?])

Let $F = [f_1, \ldots, f_m]$ a vector of m polynomials and d a non negative integer then the Macaulay matrix in degree d of F $\mathcal{M}_d^{\text{acaulay}}(F)$, is the matrix representation of

$$F^{(d)} = \left[t_j \cdot f_i \mid 1 \leqslant i \leqslant m \text{ and } t_j \in T \text{ with } \deg(t_j) \leqslant d - \deg(f_i) \right]$$

$$\mathcal{M}_d^{\text{acaulay}}(F) = M(F^{(d)}) = \begin{array}{c|ccc} & m_1 & m_2 & m_3 \\ t_1 f_1 & \cdots & \cdots & \cdots \\ t_2 f_2 & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{array}$$

Echelon form of a matrix

The basis operation is to compute a row echelon form of matrix; this will be the most costly operation.

Definition

If $M(F)$ is the matrix representation of a vector of polynomials F we denote by $\widetilde{M(F)}$ the Gaussian elimination of $M(F)$ (without pivoting the columns of the matrix).

We extend this definition to polynomials:

Definition

Let $F \subset \mathbb{K}[x_1, \ldots, x_n]$ and $<$ a monomial ordering. We denote by $\tilde{F}$ the polynomial representation of $\widetilde{M(F)}$. We say that $\tilde{F}$ is the echelon form of F (or a Gaussiant elimination) wrt $<$.

Example

The matrix representation of $F_1 = [f_3, bf_4, df_4]$ is:

$$A_1 = M(F_1) = \begin{array}{c|ccccccc} & ab & b^2 & bc & ad & bd & cd & d^2 \\ df_4 & & & & 1 & 1 & 1 & 1 \\ f_3 & 1 & & 1 & 1 & & 1 & \\ bf_4 & 1 & 1 & 1 & & 1 & & \end{array}$$

Example

The matrix representation of $F_1 = [f_3, bf_4, df_4]$ is:

$$\tilde{A}_1 = \begin{array}{c|ccccccc} & ab & b^2 & bc & ad & bd & cd & d^2 \\ df_4 & & & & 1 & 1 & 1 & 1 \\ f_3 & 1 & & 1 & & -1 & & -1 \\ bf_4 & & 1 & & & 2 & & 1 \end{array}$$

Example

The polynomial representation of:

$$\tilde{A}_1 = \begin{array}{c} \\ df_4 \\ f_3 \\ bf_4 \end{array} \begin{array}{c} \begin{array}{ccccccc} ab & b^2 & bc & ad & bd & cd & d^2 \end{array} \\ \left| \begin{array}{ccccccc} & & & 1 & 1 & 1 & 1 \\ 1 & & 1 & & -1 & & -1 \\ & 1 & & & 2 & & 1 \end{array} \right| \end{array}$$

$$\tilde{F}_1 = \begin{bmatrix} f_5 = ad + bd + cd + d^2, \\ f_6 = ab + bc - bd - d^2, \\ f_7 = b^2 + 2\,bd + d^2 \end{bmatrix}$$

Macaulay method

The idea of using linear algebra to solve polynomial systems date back to Macaulay.
Macaulay matrix is a generalization of the Sylvester matrix [**?**] (the matrix involved in the computation of the resultant of 2 polynomials).

The link between the computation of a truncated d-Gröbner basis is given by the following theorem of Lazard:

Theorem (Lazard)

If $\widetilde{F = \{f_1, \ldots, f_m\}}$ *is a set of homogeneous polynomials then* $\mathcal{M}_d^{\widetilde{\text{acaulay}}}(F)$ *is a (non reducible)* d*-Gröbner basis of* F*.*
If $\widetilde{F = \{f_1, \ldots, f_m\}}$ *is a set of polynomials, then there exists* $d > 0$ *such that* $\mathcal{M}_d^{\widetilde{\text{acaulay}}}(F)$ *is a Gröbner basis of* F*.*

Macaulay bound

Theorem (Macaulay bound)

Let $F = \{f_1, \ldots, f_m\}$ is a set of homogeneous polynomials which is a regular sequence. We define

$$D = 1 + \sum_{i=1}^{m} (deg(f_i) - 1)$$

then $\widetilde{\mathcal{M}_D^{\text{acaulay}}(F)}$ is a (non reduced) Gröbner basis of F.

Regular sequence I

We consider the Macaulay matrix of $F = [f_1, \ldots, f_m]$.
If the Macaulay matrix is singular $\longleftrightarrow$ the rows of the matrix are not independent.
Moreover, each row of the matrix is a product $t \times f$ where t is a term and $f \in F$; the linear dependence can be expressed by $\sum_{f \in F, t \in T} \lambda_{t,f}\ t\, f = 0$ or equivalently by grouping terms:

$$\sum_{i=1}^{m} g_i f_i = 0 \tag{1}$$

where g_i are polynomials in $\mathbb{K}[x_1, \ldots, x_n]$. We say that $(g_1, \ldots, g_m)$ is a syzygy. The relation (5) can be rewritten:

$$g_1 f_1 = 0 \text{ modulo } \mathrm{Id}(f_2, \ldots, f_m) \tag{2}$$

in other words it is a **zero divisor** $(\textit{if}\ g_1 \neq 0)$.

Regular sequence II

A linear system is non singular if one cannot find a non zero linear combination:

$$\sum_{i=1}^{m} \lambda_i f_i = 0 \text{ with } \lambda_i \in \mathbb{K} \tag{3}$$

For algebraic systems: it is not possible to avoid non zero relations (5) :

$$f_i f_j - f_j f_i = 0 \tag{4}$$

We say that it is a trivial syzygy.

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A linear system is non singular if one cannot find a non zero linear combination:

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For algebraic systems: it is not possible to avoid non zero relations (5) :

$$f_i f_j - f_j f_i = 0 \tag{8}$$

We say that it is a trivial syzygy.

Regular sequence III

Definition (Regular Sequence

Algebraic definition: the system $(f_1, \ldots, f_m)$ of homogeneous polynomials is **regular** if for all $i = 1, \ldots, m$ and g such that

$$g \cdot f_i \in \langle f_1, \ldots, f_{i-1} \rangle$$

then g is also in $\langle f_1, \ldots, f_{i-1} \rangle$.

Geometric definition: the system$(f_1, \ldots, f_m)$ of homogeneous polynomials is **regular** if for all $i \in \{1, \ldots, m\}$, the dimension of $\langle f_1, \ldots, f_i \rangle$ is $n - i$.
We say that the sequence $(f_1, \ldots, f_m)$ is regular.
The sequence $(f_1, \ldots, f_m)$ of affine polynomials is regular if the sequence $(f_1^h, \ldots, f_m^h)$ is regular (f_i^h is the highest homogeneous par of f_i).

Regular sequence IV

Remark

Another characterization of regular sequences: there is no relation

$$\sum_i g_i \cdot f_i = 0 \text{ with } g_i \in \mathbb{K}[x_1, \ldots, x_n]$$

except the relations induced by the trivial syzygies $f_i f_j = f_j f_i$.

Remark

From the geometric definition: there is no regular sequence when $m > n$.

Link betwen Critical Pairs and Linear Algebra Characterizations of Gröbner Bases

Characterizations of Gröbner Bases

Useful characterizations of Gröbner bases.

Definition (t-representation)

Let $P = [p_1, \ldots, p_k]$ be a finite subset of $\mathbb{K}[x_1, \ldots, x_n]$, $0 \neq f \in \mathbb{K}[x_1, \ldots, x_n]$, and $t \in T$. Assume that there exists $(g_1, \ldots, g_k) \in \mathbb{K}[x_1, \ldots, x_n]^k$ such that:

$$f = \sum_{i=1}^{k} g_i \, p_i$$

We say that it is a $t-$*representation* of f wrt P if $t \geqslant \mathrm{LT}(g_i p_i)$ for all $1 \leqslant i \leqslant k$. We denote by $f = O_P(t)$ this property.
We note $f = o_P(t)$ when there exists $t' \in T$ such that $t' < t$ and $f = O_P(t')$.

Characterizations of Gröbner Bases

Proposition

If f, g are polynomials and , t is a term, P a finite subset of polynomials, then

$$\begin{array}{llll} f = O_P(t) & g = O_P(t) & \text{implies} & f + g = O_P(t) \\ f = o_P(t) & g = o_P(t) & \text{implies} & f + g = o_P(t) \\ f = O_P(t) & u \in T & \text{implies} & u\ f = O_P(u\,t) \\ f = o_P(t) & u \in T & \text{implies} & u\,f = o_P(u\,t) \end{array}$$

Proposition (R)

If $\textsc{Reduction}(p, P) = 0$ *or* $p \xrightarrow[P]{*} 0$ *then* $p = O_P(\mathrm{LT}(p))$.

Proof.

Easy exercise. □

Characterizations of Gröbner Bases

When $f = O_G(\mathrm{LT}(f))$ we say that f has a standard representation wrt G.

Theorem

G is a Gröbner basis if and only if $\forall 0 \neq f \in \mathrm{Id}(G)$, $f = O_G(\mathrm{LT}(f))$.

Proof.

Exercice. □

and what happen when

$$f \neq O_G(\mathrm{LT}(f)) \ ?$$

Cancellation

$$f \in I = \mathrm{Id}\,(f_1, \cdots, f_m)$$

By definition:

$$f = g_1 f_1 + \cdots + g_m f_m$$

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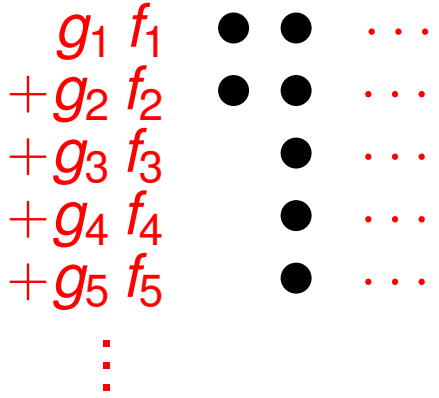
Not a unique representation !



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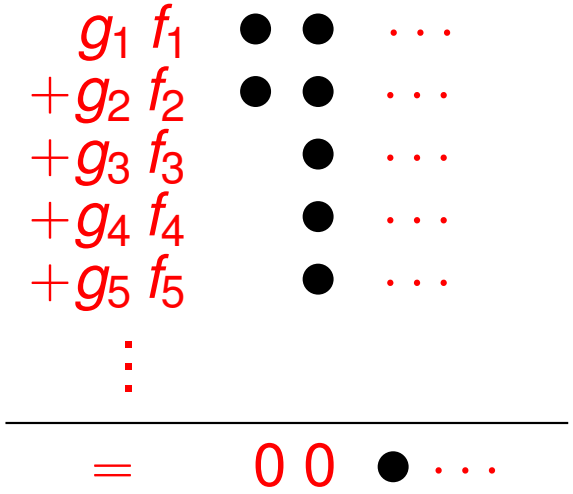

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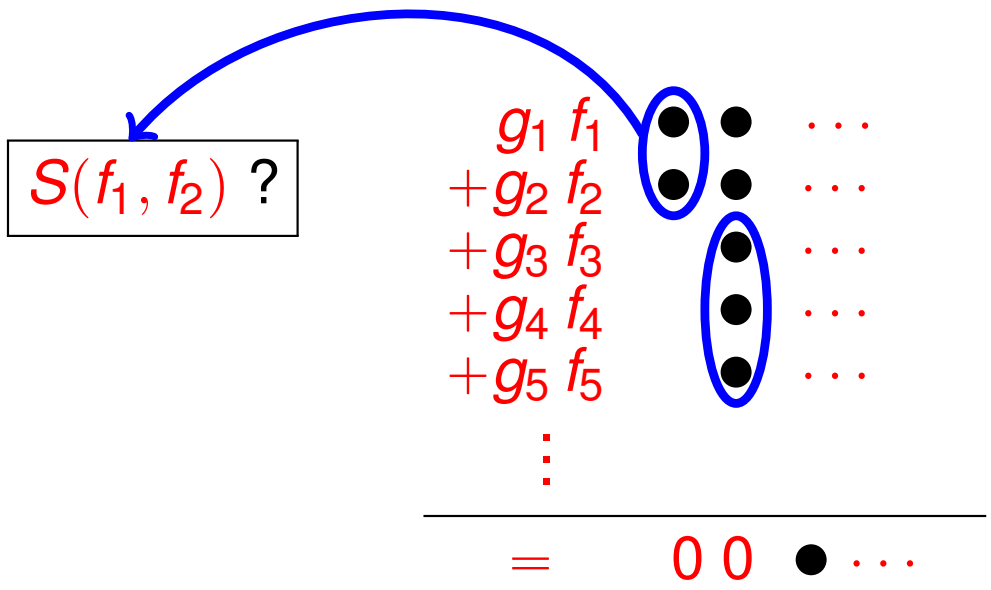

Characterizations of Gröbner Bases

Theorem

G is a Gröbner basis if and only if $\forall 0 \neq f \in \mathrm{Id}(G)$, $f = O_G(\mathrm{LT}(f))$.

Theorem

Let G be a finite subset of polynomials. If for all g_1, g_2 in G, we have $\mathrm{Spol}(g_1, g_2) = 0$ or $\mathrm{Spol}(g_1, g_2) = o_G(\mathrm{lcm}(g_1, g_2))$, then G is a Gröbner basis.

Proof.

We need to proof a lemma first . . . □

Proof of the theorem: lemma

Lemma

Let $f_1, \ldots, f_k$ *be nonzero polynomials in* $\mathbb{K}[x_1, \ldots, x_n]$ *and* $t \in T$. *Consider* $f = O_P(t) = \Sigma_{i=1}^{k} c_i \mathbf{x}^{\alpha_i} f_i$, *where* $c_i \in \mathbb{K}^*$ *such that*

$$t = \mathbf{x}^{\alpha_1} LT(f_1) = \cdots = \mathbf{x}^{\alpha_k} LT(f_k).$$

If $LT(f) < t$, *then* $k > 1$ *and* f *can be rewritten:*

$$f = \sum_{i=1}^{k-1} b_i \frac{t}{\tau_i} \operatorname{Spol}(f_i, f_{i+1}) \text{ with } b_i \in \mathbb{K}, \tag{9}$$

where $\tau_i = lcm(f_i, f_{i+1})$. *Furthermore*

$$\frac{t}{\tau_i} LT(\operatorname{Spol}(f_i, f_{i+1})) < t, \text{ for all } i = 1, \ldots, k-1.$$

Characterizations of Gröbner Bases

Corollary (Buchberger)

Let G be a finite subset of polynomials . G is a Gröbner basis if and only if $\mathrm{Spol}(f,g) \xrightarrow{*} 0$ for all $(f,g) \in G^2$.

Corollary (Buchberger)

Let G be a finite subset of polynomials . G is a Gröbner basis if and only if REDUCTION$(\mathrm{Spol}(f,g), G) = 0$ *for all $(f,g) \in G^2$.*

Proof.

Let $(f,g) \in G^2$, $f \neq g$. Put $t = \mathrm{LT}(\mathrm{Spol}(f,g)) < \mathrm{lcm}(f,g)$
If REDUCTION$(\mathrm{Spol}(f,g), G) = 0$ then from proposition (R) :
$\mathrm{Spol}(f,g) = O_G(\mathrm{LT}(\mathrm{Spol}(f,g))) = O_G(t) = o_G(\mathrm{lcm}(f,g))$and we can apply the theorem. □

Buchberger Algorithm

Very simple version of the **Buchberger** algorithm:

Algorithm (Buchberger)

Input: $\begin{cases} F = [f_1, \ldots, f_s] \text{ a list of polynomials} \\ < \text{ admissible ordering} \end{cases}$

Output: G a finite subset of $\mathbb{K}[x_1, \ldots, x_n]$.

$G := F$ and $m := s$

$P := \{(f_i, f_j) \mid 1 \leqslant i < j \leqslant m\}$ *the list of critical pairs*

while $P \neq \emptyset$ **do**

  Select and remove from P a critical pair (f, g)

  $f_{m+1} := \mathrm{Spol}(f, g)$

  $f_{m+1} :=$ REDUCTION(f_{m+1}, G)

  if $f_{m+1} \neq 0$ **then**

    $m := m + 1$

    $P := P \cup \{(f_i, f_m) \mid 1 \leqslant i < m\}$

    $G := G \cup \{f_m\}$

return G

F_4

F_4

The F_4 algorithm

Definition

A critical pairs of (f_i, f_j) is a member of $T^2 \times \mathbb{K}[x_1, \ldots, x_n] \times T \times \mathbb{K}[x_1, \ldots, x_n]$,

$$\mathrm{Pair}(f_i, f_j) := (\mathrm{lcm}_{ij}, t_i, f_i, t_j, f_j)$$

such that

$$\mathrm{lcm}(\mathrm{Pair}(f_i, f_j)) = \mathrm{lcm}_{ij} = \mathrm{LT}(t_i f_i) = \mathrm{LT}(t_j f_j) = \mathrm{lcm}(f_i, f_j)$$

Definition

We define the degree of the critical pair $p_{i,j} = \mathrm{Pair}(f_i, f_j)$, $\deg(p_{i,j})$, to be $\deg(lcm_{i,j})$. We define the following operators:

$$\mathrm{Left}(p_{i,j}) := t_i \cdot f_i \;\; et \;\; \mathrm{Right}(p_{i,j}) := t_j \cdot f_j$$

Algorithm F_4 [?] (simplified version)

Input: $\begin{cases} F \text{ is a finite subset of } \mathbb{K}[x_1, \ldots, x_n] \\ \mathcal{S}el \text{ is a function } List(Pairs) \to List(Pairs) \\ \qquad \text{such that } \mathcal{S}el(l) \neq \emptyset \text{ if } l \neq \emptyset \end{cases}$

Output: un sous ensemble fini de $\mathbb{K}[x_1, \ldots, x_n]$.

$G := F$, $\tilde{F_0^+} := F$, $d := 0$ and $P := \{\mathrm{Pair}(f, g) \mid (f, g) \in G^2 \text{ with } f \neq g\}$

while $P \neq \emptyset$ **do**

$d := d + 1$

$P_d := \mathcal{S}el(P)$

$P := P \backslash P_d$

$L_d := \mathrm{Left}(P_d) \cup \mathrm{Right}(P_d)$

$\tilde{F_d^+} :=$ REDUCTION(L_d, G)

for $h \in \tilde{F_d^+}$ **do**

$P := P \cup \{\mathrm{Pair}(h, g) \mid g \in G\}$

$G := G \cup \{h\}$

return G

We can now extend the definition of reduction of a polynomial modulo a subset of$\mathbb{K}[x_1, \ldots, x_n]$, to the reduction of a subset of $\mathbb{K}[x_1, \ldots, x_n]$ modulo another subset of $\mathbb{K}[x_1, \ldots, x_n]$:

Algorithm REDUCTION

Input: L, G finite subsets of $\mathbb{K}[x_1, \ldots, x_n]$
Output: a finite subset of $\mathbb{K}[x_1, \ldots, x_n]$ (could be empty).
$F :=$SYMBOLICPREPROCESSING(L, G)
$\tilde{F} :=$ Gaussian reduction of F wrt $<$
$\tilde{F}^+ := \left\{ f \in \tilde{F} \mid \mathrm{LT}(f) \notin \mathrm{LT}(F) \right\}$ // *the "useful" part of* $\tilde{F}$
return $\tilde{F}^+$

No arithmetic operation is used: it is a symbolic preprocessing.

Algorithm SymbolicPreprocessing

Input: L, G finite subsets of $\mathbb{K}[x_1, \ldots, x_n]$
Output: a finite subset of $\mathbb{K}[x_1, \ldots, x_n]$
$F := L$
$Done := \mathrm{LT}(F)$
while $\mathrm{T}(F) \neq Done$ **do**
    choose m an element of $\mathrm{T}(F) \backslash Done$
    $Done := Done \cup \{m\}$
    if m top réductible modulo G **then**
        exists $g \in G$ and $m' \in T$ such that $m = m' \cdot \mathrm{LT}(g)$
        $F := F \cup \{m' \cdot g\}$
return F

The SymbolicPreprocessing function is very efficient: its complexity is proportional to the size of the output (if $\#G$ is smaller than the final size of $T(F)$) [parallel implementation].

Lemma (1)

For all polynomials $p \in L_d$,we have $p \xrightarrow{G \cup \tilde{F^+}} 0$

Theorem

The F_4 algorithm computes a Gröbner basis of G in $\mathbb{K}[x_1, \ldots, x_n]$ such that $F \subseteq G$ and $Id(G) = Id(F)$.

Proof.

. . . □

Remark

If $\#Sel(l) = 1$ for all $l \neq \emptyset$ then the F_4 algorithm reduces to the Buchberger algorithm. In this case the function Sel is the equivalent of the selection strategy for the Buchberger algorithm.

Selection function

Algorithm Selection

Input: P a list of critical pairs
Output: a list of critical pairs.
$d := \min\{\deg(\mathrm{lcm}(p)) \mid p \in P\}$
$P_d := \{p \in P \mid \deg(\mathrm{lcm}(p)) = d\}$
return P_d

We call this strategy *the normal strategy for* F_4.
Hence, if the input polynomials are homogeneous, we obtain in degree d, a d Gröbner basis; $\mathcal{S}el$ selects, in the next step, all the critical pairs which are needed to compute the Gröbner basis in degree $d+1$.

Optimisations

- including Buchberger Criteria (or F_5 criterion).
- reuse **all** the rows in the reduced matrices.

Algorithm Buchberger Criteria - Implementation

$(G_{new}, P_{new}) :=$ UPDATE(G_{old}, P_{old}, h)

Input:
- a finite subset G_{old} of $\mathbb{K}[x_1, \ldots, x_n]$
- a finite subset P_{old} of critical pairs in $\mathbb{K}[x_1, \ldots, x_n]$
- $0 \neq h \in \mathbb{K}[x_1, \ldots, x_n]$

Output: a finite subset in $\mathbb{K}[x_1, \ldots, x_n]$ an updated list of critical pairs.

Algorithm F_4 algorithm (with Criteria)

Input: $\begin{cases} F \subset \mathbb{K}[x_1, \ldots, x_n] \\ \mathcal{S}el \text{ a function } \mathrm{List}(Pairs) \to \mathrm{List}(Pairs) \end{cases}$

Output: a finite subset of $\mathbb{K}[x_1, \ldots, x_n]$.

$G := \emptyset$ and$P := \emptyset$ and $d := 0$

while $F \neq \emptyset$ **do**

$f := \mathrm{first}(F)$; $F := F \backslash \{f\}$

$(G, P) :=$UPDATE(G, P, f)

while $P \neq \emptyset$ **do**

$d := d + 1$

$P_d := \mathcal{S}el(P)$; $P := P \backslash P_d$

$L_d := \mathrm{Left}(P_d) \cup \mathrm{Right}(P_d)$

$(\tilde{F_d^+}, F_d) :=$REDUCTION$(L_d, G, (F_i)_{d=1,\ldots,(d-1)})$

for $h \in \tilde{F_d^+}$ **do**

$P := P \cup \{\mathrm{Pair}(h, g) \mid g \in G\}$

$(G, P) :=$UPDATE(G, P, h)

return G

F4: step by step

Example (3 quadratic equation in $\mathbb{F}_{101}$)

Monomial ordering is DRL and the normal strategy

$$F = \left[\begin{array}{l} f_1 = {x_1}^2 + 66\,x_1\,x_2 + 4\,x_1\,x_3 + 25\,{x_2}^2 + 41\,x_2\,x_3 + 54\,{x_3}^2 + 42\,x_1 \\ \quad +87\,x_2 + 22\,x_3 + 86, \\ f_2 = {x_1}^2 + 22\,x_1\,x_2 + 38\,x_1\,x_3 + 9\,{x_2}^2 + 53\,x_2\,x_3 + 6\,{x_3}^2 + 92\,x_1 \\ \quad +61\,x_2 + 74\,x_3 + 49, \\ f_3 = {x_1}^2 + 13\,x_1\,x_2 + 86\,x_1\,x_3 + 29\,{x_2}^2 + 11\,x_2\,x_3 + 81\,{x_3}^2 + 98\,x_1 \\ \quad +67\,x_2 + 7\,x_3 + 40 \end{array} \right.$$

At the beginning $G = \{f_1\}$ and $P_1 = \{\mathrm{Pair}(f_2, f_3), \mathrm{Pair}(f_1, f_2)\}$ such that $L_1 = \{(1, f_3), (1, f_2), (1, f_1)\}$.

SymbolicPreprocessing$(L_1, G, \varnothing)$:

$F_1 = \{f_3, f_2, f_1\}$ $\quad T(F_1) = \{\boxed{x_1^2}, x_1x_2, x_1x_3, x_2^2, x_2x_3, x_3^2, x_1, x_2, x_3, 1\}$

$\boxed{x_1^2}$ is already done. All the other monomials are not reducible.

F4: step by step

Example (3 quadratic equation in $\mathbb{F}_{101}$)

At the beginning $G = \{f_1\}$ and $P_2 = \{\mathrm{Pair}(f_2, f_3), \mathrm{Pair}(f_1, f_2)\}$ such that $L_2 = \{(1, f_3), (1, f_2), (1, f_1)\}$.

SymbolicPreprocessing$(L_2, G, \varnothing)$:

$$F_2 = \{f_3, f_2, f_1\} \quad T(F_2) = \{\boxed{x_1^2}, x_1x_2, x_1x_3, x_2^2, x_2x_3, x_3^2, x_1, x_2, x_3, 1\}$$

$\boxed{x_1^2}$ is already done. All the other monomials are not reducible.

Matrix representation of $F_1 = [f_3, f_2, f_1]$ is:

$$A_1 = M(F_1) = \begin{array}{c|cccccc|c} & x_1^2 & x_1x_2 & x_2^2 & x_1x_3 & x_2x_3 & x_3^2 & \cdots \\ f_3 & 1 & 13 & 29 & 86 & 11 & 81 & \cdots \\ f_2 & 1 & 22 & 26 & 38 & 53 & 6 & \cdots \\ f_1 & 1 & 66 & 25 & 4 & 41 & 54 & \cdots \end{array}$$

F4: step by step

Example (3 quadratic equation in $\mathbb{F}_{101}$)

At the beginning $G = \{f_1\}$ and $P_2 = \{\mathrm{Pair}(f_2, f_3), \mathrm{Pair}(f_1, f_2)\}$ such that $L_2 = \{(1, f_3), (1, f_2), (1, f_1)\}$.

SymbolicPreprocessing$(L_2, G, \varnothing)$:

$$F_2 = \{f_3, f_2, f_1\} \quad T(F_2) = \{\boxed{x_1^2}, x_1x_2, x_1x_3, x_2^2, x_2x_3, x_3^2, x_1, x_2, x_3, 1\}$$

$\boxed{x_1^2}$ is already done. All the other monomials are not reducible.

$$\widetilde{A_2} = \begin{array}{c|cccccc|c} & x_1^2 & x_1x_2 & x_2^2 & x_1x_3 & x_2x_3 & x_3^2 & \cdots \\ f_6 & 0 & 0 & 1 & 28 & 19 & 79 & \cdots \\ f_5 & 0 & 1 & 0 & 12 & 2 & 5 & \cdots \\ f_1 & 1 & 66 & 4 & 25 & 41 & 54 & \cdots \end{array}$$

F4: step by step

Example (3 quadratic equation in $\mathbb{F}_{101}$)

At the beginning $G = \{f_1\}$ and $P_2 = \{\mathrm{Pair}(f_2, f_3), \mathrm{Pair}(f_1, f_2)\}$ such that $L_2 = \{(1, f_3), (1, f_2), (1, f_1)\}$.

SymbolicPreprocessing$(L_2, G, \varnothing)$:

$$F_2 = \{f_3, f_2, f_1\} \quad T(F_2) = \{\boxed{x_1^2}, x_1x_2, x_1x_3, x_2^2, x_2x_3, x_3^2, x_1, x_2, x_3, 1\}$$

$\boxed{x_1^2}$ is already done. All the other monomials are not reducible.

$$\widetilde{A_2} = \begin{array}{c|cccccc|c} & x_1^2 & x_1x_2 & x_2^2 & x_1x_3 & x_2x_3 & x_3^2 & \cdots \\ f_6 & 0 & 0 & 1 & 28 & 19 & 79 & \cdots \\ f_5 & 0 & 1 & 0 & 12 & 2 & 5 & \cdots \\ f_1 & 1 & 66 & 4 & 25 & 41 & 54 & \cdots \end{array}$$

Polynomial representation of $\widetilde{A_2}$:

F4: step by step

Example (3 quadratic equation in $\mathbb{F}_{101}$)

At the beginning $G = \{f_1\}$ and $P_2 = \{\mathrm{Pair}(f_2, f_3), \mathrm{Pair}(f_1, f_2)\}$ such that $L_2 = \{(1, f_3), (1, f_2), (1, f_1)\}$.

SymbolicPreprocessing$(L_2, G, \varnothing)$:

$F_2 = \{f_3, f_2, f_1\} \quad T(F_2) = \{\boxed{x_1^2}, x_1x_2, x_1x_3, x_2^2, x_2x_3, x_3^2, x_1, x_2, x_3, 1\}$

$\boxed{x_1^2}$ is already done. All the other monomials are not reducible.

$$\widetilde{A_2} = \begin{array}{c|cccccc|c} & x_1^2 & x_1x_2 & x_2^2 & x_1x_3 & x_2x_3 & x_3^2 & \cdots \\ f_6 & 0 & 0 & 1 & 28 & 19 & 79 & \cdots \\ f_5 & 0 & 1 & 0 & 12 & 2 & 5 & \cdots \\ f_1 & 1 & 66 & 4 & 25 & 41 & 54 & \cdots \end{array}$$

Polynomial representation of $\widetilde{A_2}$:

$f_5 = x_1\,x_2 + 12\,x_1\,x_3 + 2\,x_2\,x_3 + 55\,{x_3}^2 + 66\,x_1 + 88\,x_2 + 60\,x_3 + 92,$
$f_6 = {x_2}^2 + 28\,x_1\,x_3 + 19\,x_2\,x_3 + 79\,{x_3}^2 + 30\,x_1 + 50\,x_2 + 59\,x_3 + 46$

F4: step by step

Example (3 quadratic equation in $\mathbb{F}_{101}$)

In degree 3: $P_3 = \{\mathrm{Pair}(f_1, f_5), \mathrm{Pair}(f_5, f_6)\}$ such that
$L_3 = \{(x_2, f_1), (x_1, f_5), (x_2, f_5), (x_1, f_6)\}$.

SymbolicPreprocessing$(L_3, G, \varnothing)$:
$F_3 = \{x_2 f_1, x_1 f_5, x_2 f_5, x_1 f_6\}$
$T(F_3) = \{\boxed{x_1^2 x_2}, \boxed{x_1 x_2^2}, x_2^3, x_1 x_2 x_3, x_1 x_3^2, x_1 x_3, \ldots\}$

F4: step by step

Example (3 quadratic equation in $\mathbb{F}_{101}$)

In degree 3: $P_3 = \{\mathrm{Pair}(f_1, f_5), \mathrm{Pair}(f_5, f_6)\}$ such that
$L_3 = \{(x_2, f_1), (x_1, f_5), (x_2, f_5), (x_1, f_6)\}$.

SymbolicPreprocessing$(L_3, G, \varnothing)$:
$F_3 = \{x_2 f_1, x_1 f_5, x_2 f_5, x_1 f_6\}$
$T(F_3) = \{\boxed{x_1^2 x_2}, \boxed{x_1 x_2^2}, x_2^3, x_1 x_2 x_3, x_1 x_3^2, x_1 x_3, \ldots\}$
x_2^3 is divisible by $x_2 f_6 \longrightarrow F_3 = F_3 \cup \{x_2 f_6\}$
$x_1 x_2 x_3$ is divisible by $x_3 f_5 \longrightarrow F_3 = F_3 \cup \{x_3 f_5\}$
$\ldots$

F4: step by step

Example (3 quadratic equation in $\mathbb{F}_{101}$)

In degree 3: $P_3 = \{\mathrm{Pair}(f_1, f_5), \mathrm{Pair}(f_5, f_6)\}$ such that
$L_3 = \{(x_2, f_1), (x_1, f_5), (x_2, f_5), (x_1, f_6)\}$.

SymbolicPreprocessing$(L_3, G, \varnothing)$:
$F_3 = \{x_2 f_1, x_1 f_5, x_2 f_5, x_1 f_6\}$
$T(F_3) = \{\boxed{x_1^2 x_2}, \boxed{x_1 x_2^2}, x_2^3, x_1 x_2 x_3, x_1 x_3^2, x_1 x_3, \ldots\}$
x_2^3 is divisible by $x_2 f_6 \longrightarrow F_3 = F_3 \cup \{x_2 f_6\}$
$x_1 x_2 x_3$ is divisible by $x_3 f_5 \longrightarrow F_3 = F_3 \cup \{x_3 f_5\}$
$\ldots$
$F_3 = [x_2\, f_1, x_1\, f_5, x_2\, f_5, x_1\, f_6, x_2\, f_6, x_3\, f_5, x_3\, f_6, f_5, f_6, x_3\, f_1, f_1]$

F4: step by step

Example (3 quadratic equation in $\mathbb{F}_{101}$)

x_2^3 is divisible by $x_2 f_6 \longrightarrow F_3 = F_3 \cup \{x_2 f_6\}$
$x_1 x_2 x_3$ is divisible by $x_3 f_5 \longrightarrow F_3 = F_3 \cup \{x_3 f_5\}$
...

$$F_3 = [x_2\, f_1, x_1\, f_5, x_2\, f_5, x_1\, f_6, x_2\, f_6, x_3\, f_5, x_3\, f_6, f_5, f_6, x_3\, f_1, f_1]$$

$$A_3 = \begin{array}{r|cccccccccccccccc}
f_6 & & & & & & & & & & & 1 & 28 & 19 & 79 & 30 & 50 \\
f_5 & & & & & & & & & & 1 & 0 & 12 & 2 & 55 & 66 & 88 \\
. & & & & & & & & & 1 & 66 & 25 & 4 & 41 & 54 & 42 & 87 \\
. & & & & & 1 & 28 & 19 & 79 & 0 & 0 & 0 & 30 & 50 & 59 & 0 & 0 \\
. & & & & 1 & 0 & 12 & 2 & 55 & 0 & 0 & 0 & 66 & 88 & 60 & 0 & 0 \\
. & & & 1 & 66 & 25 & 4 & 41 & 54 & 0 & 0 & 0 & 42 & 87 & 22 & 0 & 0 \\
. & & 1 & 0 & 28 & 19 & 0 & 79 & 0 & 0 & 30 & 50 & 0 & 59 & 0 & 0 & 46 \\
x_2 f_5 & 1 & 0 & 0 & 12 & 2 & 0 & 55 & 0 & 0 & 66 & 88 & 0 & 60 & 0 & 0 & 92 \\
f_{10} & 1 & 0 & 28 & 19 & 0 & 79 & 0 & 0 & 30 & 50 & 0 & 59 & 0 & 0 & 46 & 0 \\
x_2 f_1 & 1 & 66 & 25 & 0 & 4 & 41 & 0 & 54 & 0 & 0 & 42 & 87 & 0 & 22 & 0 & 0 & 86 \\
f_8 & 1 & 0 & 0 & 12 & 2 & 0 & 55 & 0 & 0 & 66 & 88 & 0 & 60 & 0 & 0 & 92 & 0
\end{array}$$

$$f_{10} = x_1\, {x_3}^2 + 23\, {x_3}^3 + 77\, x_1\, x_3 + 66\, x_2\, x_3 + 84\, {x_3}^2 + 48\, x_1 + 38\, x_2 + 44\, x_3 + 68$$
$$f_8 = x_2\, {x_3}^2 + 98\, {x_3}^3 + 60\, x_1\, x_3 + 34\, x_2\, x_3 + 85\, {x_3}^2 + 65\, x_1 + 9\, x_2 + 74\, x_3 + 28$$

F4: step by step

Example (Cyclic 4)

Monomial ordering is DRL and the normal strategy

$$F = \left[\begin{array}{c} f_1 = abcd - 1, f_2 = abc + abd + acd + bcd, \\ f_3 = ab + bc + ad + cd, f_4 = a + b + c + d \end{array} \right]$$

At the beginning $G = \{f_4\}$ and $P_1 = \{\mathrm{Pair}(f_3, f_4)\}$ such that $L_1 = \{(1, f_3), (b, f_4)\}$.

Example (Cyclic 4)

Monomial ordering is DRL and the normal strategy

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At the beginning $G = \{f_4\}$ and $P_1 = \{\mathrm{Pair}(f_3, f_4)\}$ such that $L_1 = \{(1, f_3), (b, f_4)\}$.

SymbolicPreprocessing$(L_1, G, \emptyset)$:

$F_1 = \{f_3, b\, f_4\}$ $\quad T(F_1) = \{\boxed{ab}, ad, b^2, bc, bd, cd\}$

$\boxed{ab}$ is already done.

Example (Cyclic 4)

Monomial ordering is DRL and the normal strategy

$$F = \left[\begin{array}{c} f_1 = abcd - 1, f_2 = abc + abd + acd + bcd, \\ f_3 = ab + bc + ad + cd, f_4 = a + b + c + d \end{array} \right]$$

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SymbolicPreprocessing$(L_1, G, \varnothing)$:

$F_1 = \{f_3, b\, f_4\}$ $\quad T(F_1) = \{\boxed{ab}, \boxed{ad}, b^2, bc, bd, cd\}$

Example (Cyclic 4)

Monomial ordering is DRL and the normal strategy

$$F = \left[\begin{array}{c} f_1 = abcd - 1, f_2 = abc + abd + acd + bcd, \\ f_3 = ab + bc + ad + cd, f_4 = a + b + c + d \end{array} \right]$$

At the beginning $G = \{f_4\}$ and $P_1 = \{\mathrm{Pair}(f_3, f_4)\}$ such that $L_1 = \{(1, f_3), (b, f_4)\}$.

SymbolicPreprocessing$(L_1, G, \varnothing)$:

$F_1 = \{f_3, b\, f_4\}$ $\quad T(F_1) = \{\boxed{ab}, \boxed{ad}, b^2, bc, bd, cd\}$

ad is top reducible by $f_4 \in G$!

Example (Cyclic 4)

Monomial ordering is DRL and the normal strategy

$$F = \left[\begin{array}{c} f_1 = abcd - 1, f_2 = abc + abd + acd + bcd, \\ f_3 = ab + bc + ad + cd, f_4 = a + b + c + d \end{array} \right]$$

At the beginning $G = \{f_4\}$ and $P_1 = \{\mathrm{Pair}(f_3, f_4)\}$ such that $L_1 = \{(1, f_3), (b, f_4)\}$.

SymbolicPreprocessing$(L_1, G, \varnothing)$:

$F_1 = \{f_3, b\,f_4, df_4\}$ $\quad T(F_1) = \{\boxed{ab}, \boxed{ad}, b^2, bc, bd, cd, d^2\}$

Example (Cyclic 4)

Monomial ordering is DRL and the normal strategy

$$F = \left[\begin{array}{c} f_1 = abcd - 1, f_2 = abc + abd + acd + bcd, \\ f_3 = ab + bc + ad + cd, f_4 = a + b + c + d \end{array} \right]$$

At the beginning $G = \{f_4\}$ and $P_1 = \{\mathrm{Pair}(f_3, f_4)\}$ such that $L_1 = \{(1, f_3), (b, f_4)\}$.

SymbolicPreprocessing$(L_1, G, \varnothing)$:

$F_1 = \{f_3, b\,f_4, df_4\}$ $\quad T(F_1) = \{\boxed{ab}, \boxed{ad}, \boxed{b^2}, bc, bd, cd, d^2\}$

Example (Cyclic 4)

Monomial ordering is DRL and the normal strategy

$$F = \left[\begin{array}{c} f_1 = abcd - 1, f_2 = abc + abd + acd + bcd, \\ f_3 = ab + bc + ad + cd, f_4 = a + b + c + d \end{array} \right]$$

At the beginning $G = \{f_4\}$ and $P_1 = \{\mathrm{Pair}(f_3, f_4)\}$ such that $L_1 = \{(1, f_3), (b, f_4)\}$.

SymbolicPreprocessing$(L_1, G, \emptyset)$:

$F_1 = \{f_3, b\,f_4, df_4\}$ $\quad T(F_1) = \{\boxed{ab}, \boxed{ad}, \boxed{b^2}, bc, bd, cd, d^2\}$

b^2 is not reducible by G

Example (Cyclic 4)

Monomial ordering is DRL and the normal strategy

$$F = \left[\begin{array}{c} f_1 = abcd - 1, f_2 = abc + abd + acd + bcd, \\ f_3 = ab + bc + ad + cd, f_4 = a + b + c + d \end{array} \right]$$

At the beginning $G = \{f_4\}$ and $P_1 = \{\mathrm{Pair}(f_3, f_4)\}$ such that $L_1 = \{(1, f_3), (b, f_4)\}$.

SymbolicPreprocessing$(L_1, G, \varnothing)$:

$F_1 = \{f_3, b\,f_4, df_4\}$ $\quad T(F_1) = \{\boxed{ab}, \boxed{ad}, \boxed{b^2}, \boxed{bc}, \boxed{bd}, \boxed{cd}, \boxed{d^2}\}$

Example (Cyclic 4)

Monomial ordering is DRL and the normal strategy

$$F = \left[\begin{array}{c} f_1 = abcd - 1, f_2 = abc + abd + acd + bcd, \\ f_3 = ab + bc + ad + cd, f_4 = a + b + c + d \end{array} \right]$$

At the beginning $G = \{f_4\}$ and $P_1 = \{\mathrm{Pair}(f_3, f_4)\}$ such that $L_1 = \{(1, f_3), (b, f_4)\}$.

SymbolicPreprocessing $(L_1, G, \varnothing)$ returns

$$F_1 = [f_3, bf_4, df_4].$$

Example (Cyclic 4)

Matrix representation of $F_1 = [f_3, bf_4, df_4]$ is:

$$A_1 = M(F_1) = \begin{array}{c|ccccccc} & ab & b^2 & bc & ad & bd & cd & d^2 \\ df_4 & & & & 1 & 1 & 1 & 1 \\ f_3 & 1 & & 1 & 1 & & 1 & \\ bf_4 & 1 & 1 & 1 & & 1 & & \end{array}$$

Example (Cyclic 4)

Gaussian reduction of A_1 is:

$$\widetilde{A_1} = \begin{array}{c|ccccccc} & ab & b^2 & bc & ad & bd & cd & d^2 \\ df_4 & & & & 1 & 1 & 1 & 1 \\ f_3 & 1 & & 1 & & -1 & & -1 \\ bf_4 & & 1 & & & 2 & & 1 \end{array}$$

Example (Cyclic 4)

$$\widetilde{A_1} = \begin{array}{c|ccccccc} & ab & b^2 & bc & ad & bd & cd & d^2 \\ df_4 & & & & 1 & 1 & 1 & 1 \\ f_3 & 1 & & 1 & & -1 & & -1 \\ bf_4 & & 1 & & & 2 & & 1 \end{array}$$

$$\tilde{F}_1 = \begin{bmatrix} f_5 = ad + bd + cd + d^2, \\ f_6 = ab + bc - bd - d^2, \\ f_7 = b^2 + 2\,bd + d^2 \end{bmatrix}$$

Example (Cyclic 4)

$\tilde{F_1} = [f_5 = ad + bd + cd + d^2,$
$f_6 = ab + bc - bd - d^2,$
$f_7 = b^2 + 2\,bd + d^2]$

and since $ab, ad \in \mathrm{LT}(F_1)$ we have
$\tilde{F_1}_+ = [f_7]$
and now $G = \{f_4, f_7\}$.

Example (Cyclic 4)

For the next step we have to consider $P_2 = \{\mathrm{Pair}(f_2, f_4)\}$
hence $L_2 = \{(1, f_2), (bc, f_4)\}$ and $\mathcal{F} = \{F_1\}$.

Example (Cyclic 4)

$L_2 = \{(1, f_2), (bc, f_4)\}$ et $\mathcal{F} = \{F_1\}$.
In SymbolicPreprocessing we can try to simplify the products $1 \cdot f_2$ and $bc \cdot f_4$ using the previous computations:

For instance $\mathrm{LT}(bc\, f_4) = abc = \mathrm{LT}(c\, f_6)$ and so instead of $bc \cdot f_4$ we can consider $c \cdot f_6$.

Example (Cyclic 4)

For the next step we have to consider $P_2 = \{\mathrm{Pair}(f_2, f_4)\}$
hence $L_2 = \{(1, f_2), (bc, f_4)\}$ and $\mathcal{F} = \{F_1\}$.
SymbolicPreprocessing

$F_2 = \{f_2, c\,f_6\}$ $\quad T(F_2) = \{\boxed{abc}, bc^2, abd, acd, bcd, cd^2\}$

Example (Cyclic 4)

For the next step we have to consider $P_2 = \{\mathrm{Pair}(f_2, f_4)\}$
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SymbolicPreprocessing

$F_2 = \{f_2, cf_6\}$ $\quad T(F_2) = \{\boxed{abc}, bc^2, \boxed{abd}, acd, bcd, cd^2\}$

Example (Cyclic 4)

$\tilde{F_1} = [f_5 = ad + bd + cd + d^2, f_6 = ab + bc - bd - d^2, f_7 = b^2 + 2\,bd + d^2]$

For the next step we have to consider $P_2 = \{\mathrm{Pair}(f_2, f_4)\}$
hence $L_2 = \{(1, f_2), (bc, f_4)\}$ and $\mathcal{F} = \{F_1\}$.
SymbolicPreprocessing

$F_2 = \{f_2, cf_6\} \quad T(F_2) = \{\boxed{abc}, bc^2, \boxed{abd}, acd, bcd, cd^2\}$

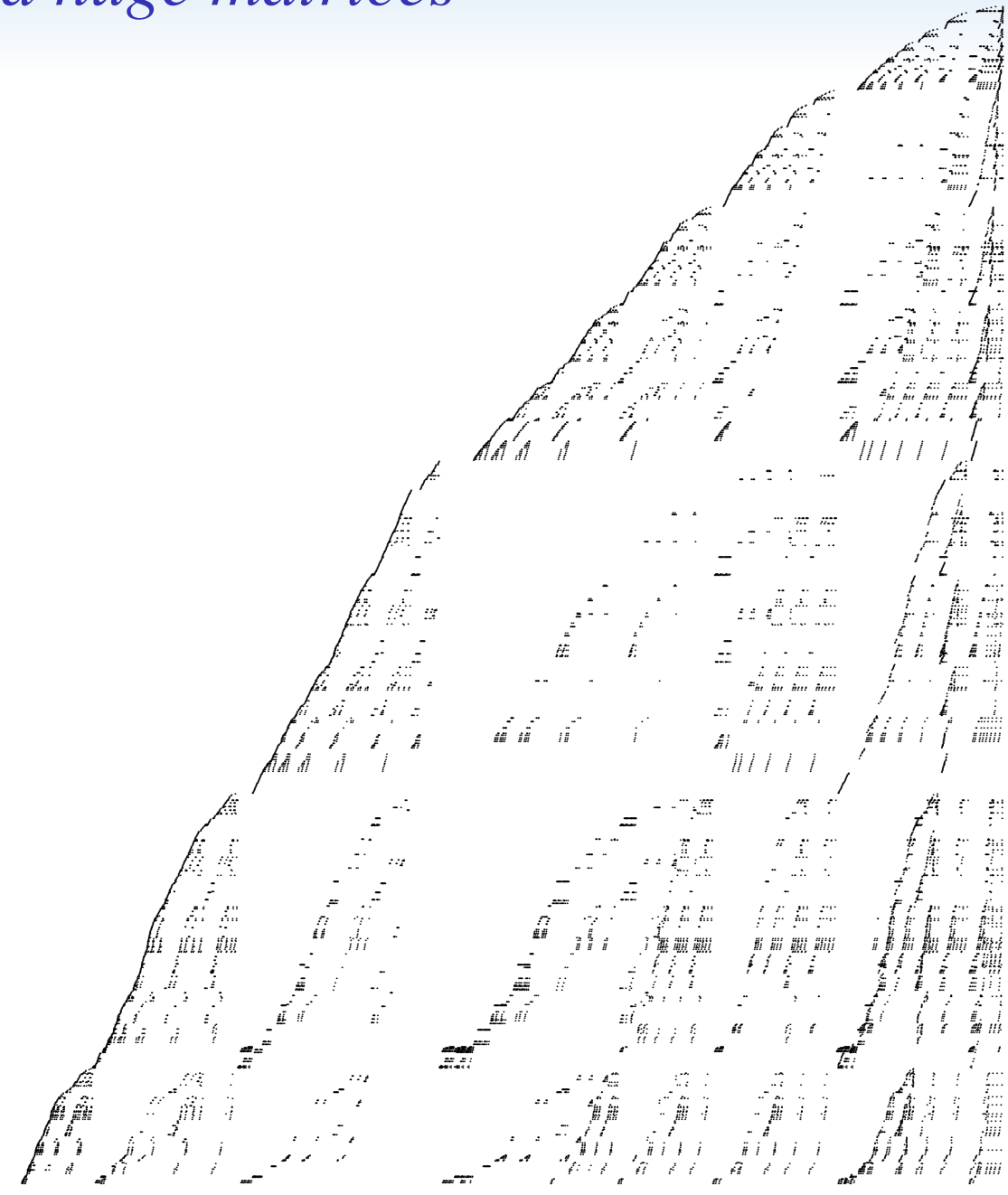
abd is reducible by $bd\,f_4$ but also by $b\,f_5$!

Optimisations

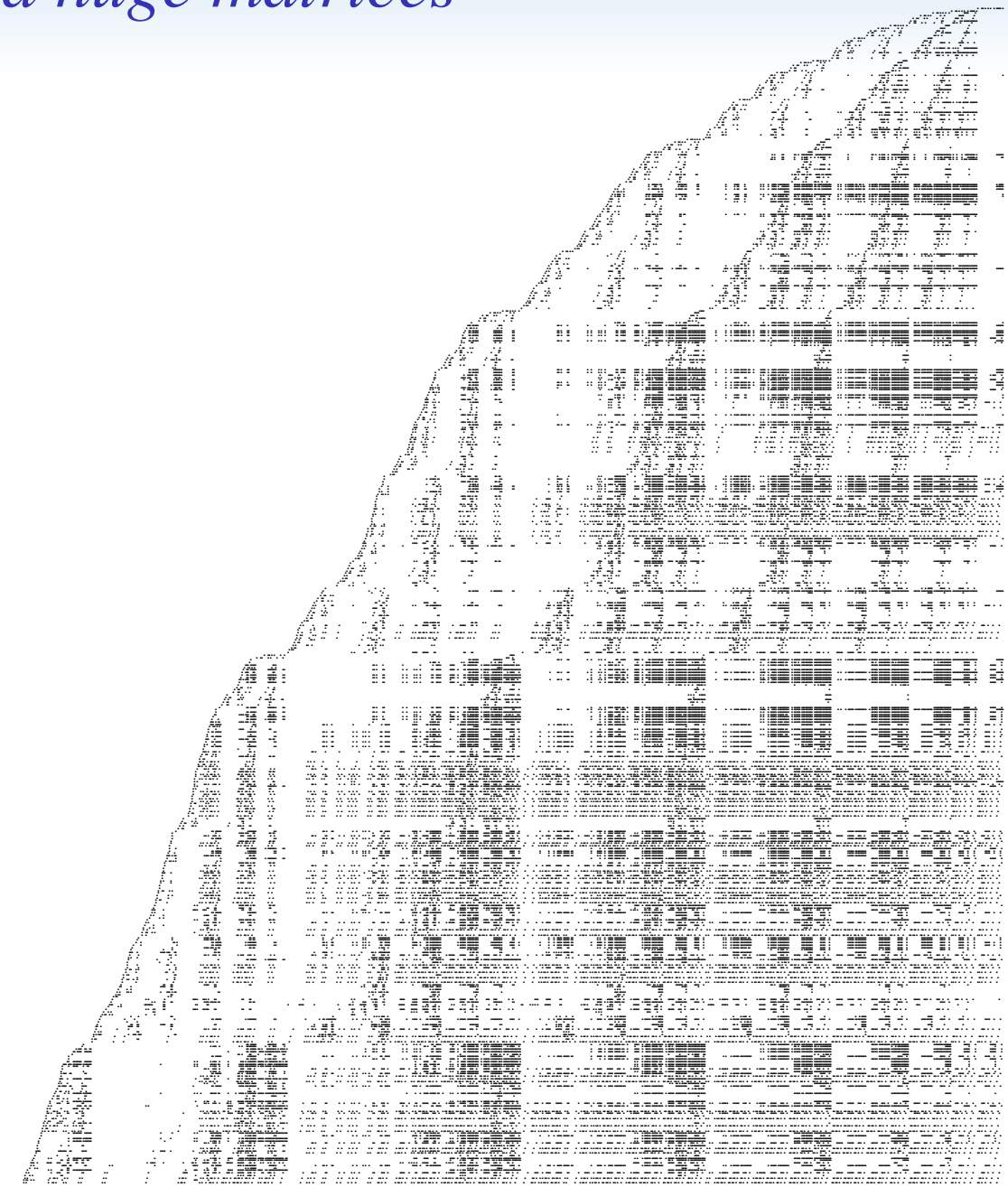
- including Buchberger Criteria (or F_5 criterion).

- reuse **all** the rows in the reduced matrices.
- Improve the linear algebra step (dedicated algorithms, matrix compression, . . .)

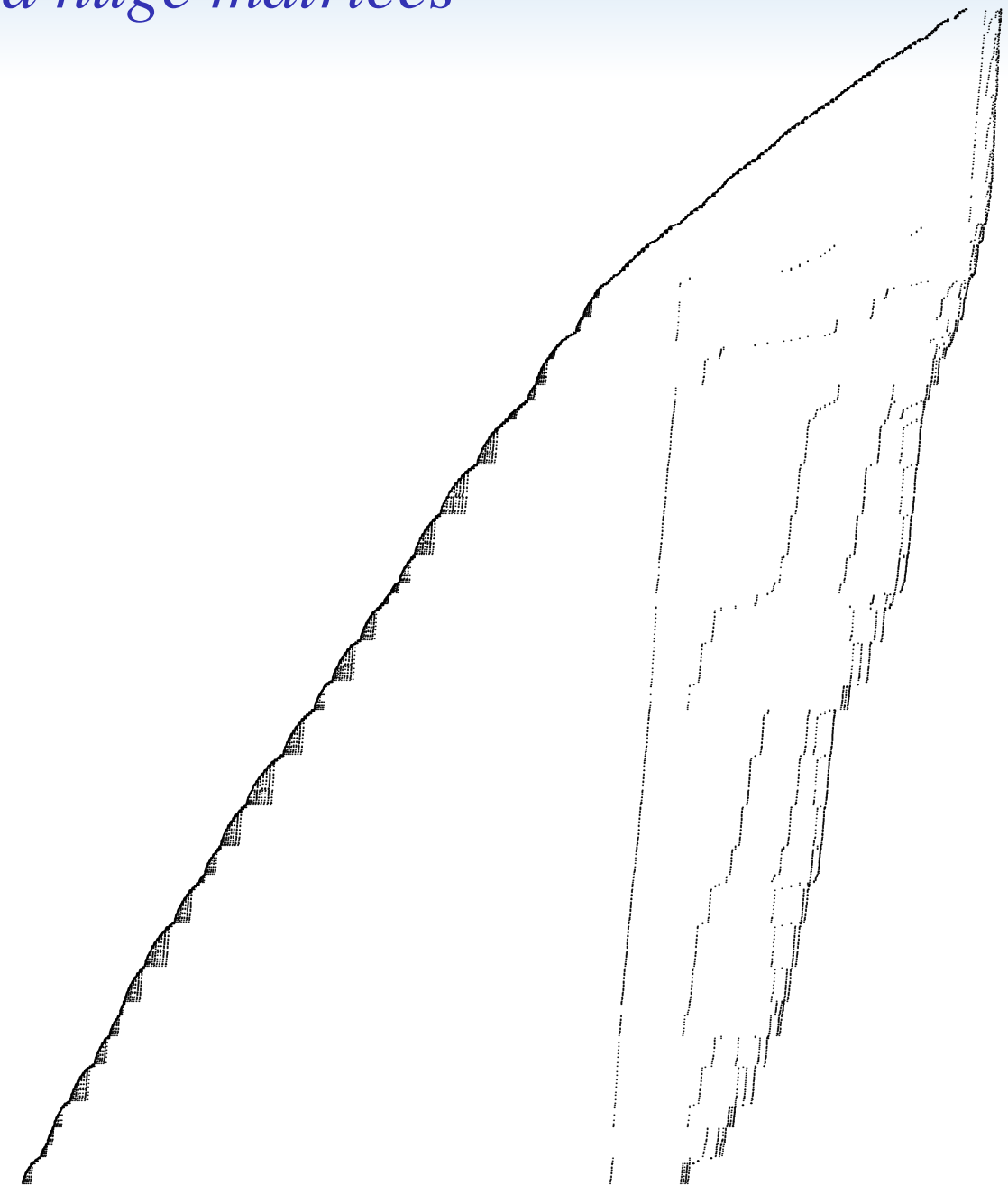
F_4 generated huge matrices



F_4 generated huge matrices



F_4 *generated huge matrices*



Need to compress the matrices !

(i) Compression bitmap: denote by

$$j_1, j_2, j_3, \ldots$$

the position of the non zero elements in the matrix, then

$$\sum_k 2^{j_k - 1}$$

is the corresponding bitmap.
This efficient but the reduction factor is not big (constant factor).

Compress the matrices

(ii) Another idea is to consider the differences (Lempel-Ziv coding):

$$\boxed{j_1}\boxed{j_2 - j_1}\boxed{j_3 - j_2}\cdots$$

when the difference $j_k - j_{k-1}$ is small (< 128), $\longrightarrow$ we can use one byte to store the result.
This method is is more efficient wrt the memory usage and only slightly slower (10%).

Algorithms

Algorithms: for *computing* Gröbner bases.

- Buchberger (1965,1979,1985)
  - ☞ First and Second Criteria
- F_4 using linear algebra (1999) (strategies)
- F_5 no reduction to zero (2002)
  - Today $\longrightarrow$ simple matrix F_5 algorithm
- Signature-based Gröbner computations (2008-. . .)

F_5 *algorithm*

- Goal: avoid useless reduction to 0
  generate full rank matrices
- Incremental algorithm

$$(f_1) + G_{\text{prev}}$$

- We have to explain: new F_5 criterion

F_5 *an example I*

We consider the following example: (b is a parameter):

$$\mathcal{S}_b \begin{cases} f_3 = x^2 + 18\,xy + 19\,y^2 + 8\,xz + 5\,yz + 7\,z^2 \\ f_2 = 3\,x^2 + (7 + b)\,x\,y + 22\,x\,z + 11\,yz + 22\,z^2 + 8\,y^2 \\ f_1 = 6\,x^2 + 12\,xy + 4\,y^2 + 14\,xz + 9\,yz + 7\,z^2 \end{cases}$$

For now we assume that $b = 0$
With Buchberger $x > y > z$:

- 5 useless reductions
- 5 useful pairs

F_5 *an example II*

We proceed degree by degree.

$$A_2 = \begin{array}{c|cccccc} & x^2 & x\,y & y^2 & x\,z & y\,z & z^2 \\ f_3 & 1 & 18 & 19 & 8 & 5 & 7 \\ f_2 & 3 & 7 & 8 & 22 & 11 & 22 \\ f_1 & 6 & 12 & 4 & 14 & 9 & 7 \end{array}$$

$$\widetilde{A_2} = \begin{array}{c|cccccc} & x^2 & x\,y & y^2 & x\,z & y\,z & z^2 \\ f_3 & 1 & 18 & 19 & 8 & 5 & 7 \\ f_2 & & 1 & 3 & 2 & 4 & -1 \\ f_1 & & & 1 & -11 & -3 & -5 \end{array}$$

"new" polynomials $f_4 = xy + 4\,yz + 2\,xz + 3\,y^2 - z^2$ and $f_5 = y^2 - 11\,xz - 3\,yz - 5\,z^2$

F_5 *an example III*

$$
\begin{aligned}
f_3 &= x^2 + 18\,xy + 19\,y^2 + 8\,xz + 5\,yz + 7\,z^2 \\
f_2 &= 3\,x^2 + 7x\,y + 22\,x\,z + 11\,yz + 22\,z^2 + 8\,y^2 \\
f_1 &= 6\,x^2 + 12\,xy + 4\,y^2 + 14\,xz + 9\,yz + 7\,z^2 \\
f_4 &= xy + 4\,yz + 2\,xz + 3\,y^2 - z^2 \\
f_5 &= y^2 - 11\,xz - 3\,yz - 5\,z^2
\end{aligned}
$$

$$
\begin{aligned}
f_2 &\longrightarrow f_4 \\
f_1 &\longrightarrow f_5
\end{aligned}
$$

Degree 3 (first try)

$$
\begin{aligned}
f_3 &= x^2 + 18\,xy + 19\,y^2 + 8\,xz + 5\,yz + 7\,z^2 \\
f_2 &= 3\,x^2 + 7x\,y + 22\,x\,z + 11\,yz + 22\,z^2 + 8\,y^2 \\
f_1 &= 6\,x^2 + 12\,xy + 4\,y^2 + 14\,xz + 9\,yz + 7\,z^2 \\
f_4 &= xy + 4\,yz + 2\,xz + 3\,y^2 - z^2 \\
f_5 &= y^2 - 11\,xz - 3\,yz - 5\,z^2
\end{aligned}
$$

and

$$
\begin{aligned}
f_2 &\longrightarrow f_4 \\
f_1 &\longrightarrow f_5
\end{aligned}
$$

Degree 3 (first try)

$$
A_3 := \begin{array}{c} \\ zf_3 \\ yf_3 \\ xf_3 \\ zf_2 \\ yf_2 \\ xf_2 \\ zf_1 \\ yf_1 \\ xf_1 \end{array}
\begin{array}{c}
\begin{array}{ccccccc} x^3 & x^2y & xy^2 & y^3 & x^2z & \dots \end{array} \\
\begin{pmatrix}
0 & 0 & 0 & 0 & 1 & \dots \\
0 & 1 & 18 & 19 & 0 & \dots \\
1 & 18 & 19 & 0 & 8 & \dots \\
0 & 0 & 0 & 0 & 3 & \dots \\
0 & 3 & 7 & 8 & 0 & \dots \\
3 & 7 & 8 & 0 & 22 & \dots \\
0 & 0 & 0 & 0 & 6 & \dots \\
0 & 6 & 12 & 4 & 0 & \dots \\
6 & 12 & 4 & 0 & 14 & \dots
\end{pmatrix}
\end{array}
$$

Degree 3 (first try)

$$
A_3 := \begin{array}{c} \\ zf_3 \\ yf_3 \\ xf_3 \\ zf_2 \\ yf_2 \\ xf_2 \\ zf_1 \\ yf_1 \\ xf_1 \end{array}
\begin{array}{c}
\begin{array}{ccccccc} x^3 & x^2y & xy^2 & y^3 & x^2z & \dots \end{array} \\
\begin{pmatrix}
0 & 0 & 0 & 0 & \textcircled{1} & \dots \\
0 & 1 & 18 & 19 & 0 & \dots \\
1 & 18 & 19 & 0 & 8 & \dots \\
0 & 0 & 0 & 0 & 3 & \dots \\
0 & 3 & 7 & 8 & 0 & \dots \\
3 & 7 & 8 & 0 & 22 & \dots \\
0 & 0 & 0 & 0 & 6 & \dots \\
0 & 6 & 12 & 4 & 0 & \dots \\
6 & 12 & 4 & 0 & 14 & \dots
\end{pmatrix}
\end{array}
$$

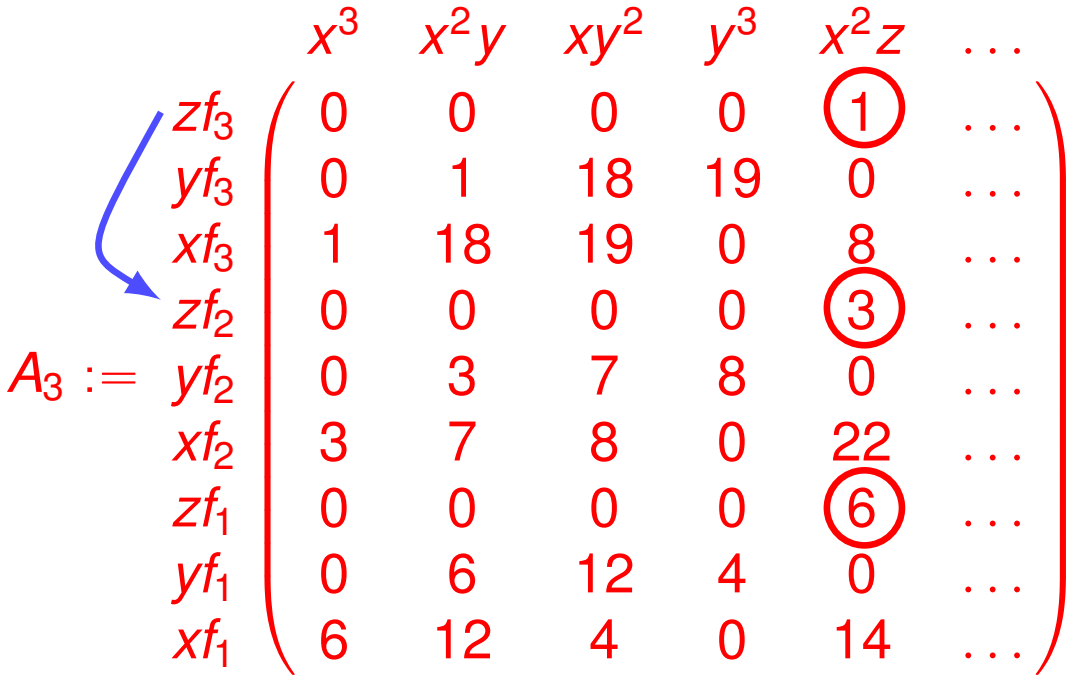
Degree 3 (first try)

$$
A_3 := \begin{array}{c} \\ zf_3 \\ yf_3 \\ xf_3 \\ zf_2 \\ yf_2 \\ xf_2 \\ zf_1 \\ yf_1 \\ xf_1 \end{array}
\begin{array}{c}
\begin{array}{ccccccc} x^3 & x^2y & xy^2 & y^3 & x^2z & \dots \end{array} \\
\begin{pmatrix}
0 & 0 & 0 & 0 & \textcircled{1} & \dots \\
0 & 1 & 18 & 19 & 0 & \dots \\
1 & 18 & 19 & 0 & 8 & \dots \\
0 & 0 & 0 & 0 & \textcircled{3} & \dots \\
0 & 3 & 7 & 8 & 0 & \dots \\
3 & 7 & 8 & 0 & 22 & \dots \\
0 & 0 & 0 & 0 & 6 & \dots \\
0 & 6 & 12 & 4 & 0 & \dots \\
6 & 12 & 4 & 0 & 14 & \dots
\end{pmatrix}
\end{array}
$$

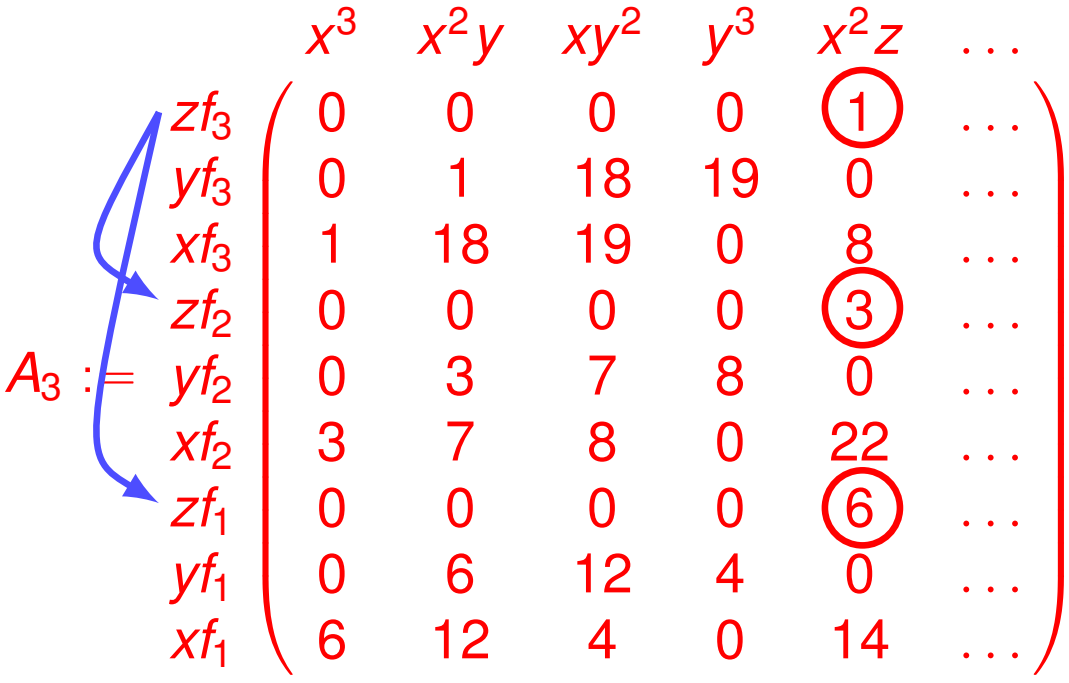
Degree 3 (first try)

$$
A_3 := \begin{array}{c} \\ zf_3 \\ yf_3 \\ xf_3 \\ zf_2 \\ yf_2 \\ xf_2 \\ zf_1 \\ yf_1 \\ xf_1 \end{array}
\begin{array}{c}
\begin{array}{ccccccc} x^3 & x^2y & xy^2 & y^3 & x^2z & \dots \end{array} \\
\begin{pmatrix}
0 & 0 & 0 & 0 & \textcircled{1} & \dots \\
0 & 1 & 18 & 19 & 0 & \dots \\
1 & 18 & 19 & 0 & 8 & \dots \\
0 & 0 & 0 & 0 & \textcircled{3} & \dots \\
0 & 3 & 7 & 8 & 0 & \dots \\
3 & 7 & 8 & 0 & 22 & \dots \\
0 & 0 & 0 & 0 & 6 & \dots \\
0 & 6 & 12 & 4 & 0 & \dots \\
6 & 12 & 4 & 0 & 14 & \dots
\end{pmatrix}
\end{array}
$$

Degree 3 (first try)



Degree 3 (first try)



Degree 3 (first try)

Already
Done !

$f_2 \longrightarrow f_4$

$f_1 \longrightarrow f_5$

$$A_3 := \begin{array}{c} \\ zf_3 \\ yf_3 \\ xf_3 \\ zf_2 \\ yf_2 \\ xf_2 \\ zf_1 \\ yf_1 \\ xf_1 \end{array} \begin{array}{c} \begin{array}{ccccccc} x^3 & x^2y & xy^2 & y^3 & x^2z & \dots \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 & \textcircled{1} & \dots \\ 0 & 1 & 18 & 19 & 0 & \dots \\ 1 & 18 & 19 & 0 & 8 & \dots \\ 0 & 0 & 0 & 0 & \textcircled{3} & \dots \\ 0 & 3 & 7 & 8 & 0 & \dots \\ 3 & 7 & 8 & 0 & 22 & \dots \\ 0 & 0 & 0 & 0 & \textcircled{6} & \dots \\ 0 & 6 & 12 & 4 & 0 & \dots \\ 6 & 12 & 4 & 0 & 14 & \dots \end{pmatrix} \end{array}$$

Degree 3

$$
A_3 := \begin{array}{c} \\ zf_3 \\ yf_3 \\ xf_3 \\ zf_4 \\ yf_4 \\ xf_4 \\ zf_5 \\ yf_5 \\ xf_5 \end{array}
\begin{array}{c}
\begin{array}{ccccccccccc} x^3 & & x^2y & & xy^2 & & y^3 & & x^2z & & xyz \end{array} \\
\begin{pmatrix}
 & & & & & & & & 1 & & 18 \\
 & & 1 & & 18 & & 19 & & 0 & & 8 \\
1 & & 18 & & 19 & & 0 & & 8 & & 5 \\
 & & & & & & & & & & 1 \\
 & & & & 1 & & 3 & & 0 & & 2 \\
 & & 1 & & 3 & & 0 & & 2 & & 4 \\
 & & & & & & & & & & \\
 & & & & & & 1 & & 0 & & 12 \\
 & & & & 1 & & 0 & & 12 & & 20
\end{pmatrix}
\end{array}
\begin{array}{c}
\begin{array}{ccccccccc} y^2z & & xz^2 & & yz^2 & & z^3 \end{array} \\
\begin{pmatrix}
19 & & 8 & & 5 & & 7 \\
5 & & 0 & & 7 & & 0 \\
0 & & 7 & & 0 & & 0 \\
3 & & 2 & & 4 & & 22 \\
4 & & 0 & & 22 & & 0 \\
0 & & 22 & & 0 & & 0 \\
1 & & 12 & & 20 & & 18 \\
20 & & 0 & & 18 & & 0 \\
0 & & 18 & & 0 & & 0
\end{pmatrix}
\end{array}
$$

Degree 3

$$
\tilde{A}_3 := \begin{array}{c} \\ xf_3 \\ yf_3 \\ yf_2 \\ xf_2 \\ zf_3 \\ zf_2 \\ zf_1 \\ yf_1 \\ xf_1 \end{array}
\begin{array}{c}
\begin{array}{cccccccccc} x^3 & x^2y & xy^2 & y^3 & x^2z & xyz & y^2z & xz^2 & yz^2 & z^3 \end{array} \\
\left(\begin{array}{cccccccccc}
1 & 18 & 19 & 0 & 8 & 5 & 0 & 7 & 0 & 0 \\
 & 1 & 18 & 19 & 0 & 8 & 5 & 0 & 7 & 0 \\
 & & 1 & 3 & 0 & 2 & 4 & 0 & 22 & 0 \\
 & & & 1 & 0 & 0 & 8 & 1 & 18 & 15 \\
 & & & & 1 & 18 & 19 & 8 & 5 & 7 \\
 & & & & & 1 & 3 & 2 & 4 & 22 \\
 & & & & & & 1 & 12 & 20 & 18 \\
 & & & & & & & 1 & 11 & 13 \\
 & & & & & & & & 1 & 18
\end{array}\right)
\end{array}
$$

Degree 3

Summary: we have constructed 3 new polynomials

$$f_6 = y^3 + 8\,y^2z + xz^2 + 18\,yz^2 + 15\,z^3$$
$$f_7 = xz^2 + 11\,yz^2 + 13\,z^3$$
$$f_8 = yz^2 + 18\,z^3$$

And we have the linear equivalences:

$$x\,f_2 \leftrightarrow x\,f_4 \leftrightarrow f_6$$
$$f_4 \longrightarrow f_2$$

Degree 4

The matrix whose rows are

$$x^2 f_i, x\, y f_i, y^2 f_i, x\, z f_i, y\, z f_i, z^2 f_i, \;\; i = 1, 2, 3$$

is not full rank !

Why ? (1)

$6 \times 3 = 18$ rows
$x^4, x^3\,y, \ldots, y\,z^3, z^4$ 15 columns

Why ? (1)

$6 \times 3 =$ 18 rows

$x^4, x^3\,y, \ldots, y\,z^3, z^4$ 15 columns

Simple linear algebra theorem: 3 useless row (but which ones ?)

Trivial relations

$$f_2\, f_3 - f_3\, f_2 = 0$$

can be rewritten

$$\begin{aligned} &3\,x^2\, f_3 + (7 + \ b)\, xy\, f_3 + 8\, y^2\, f_3 + 22\, xz\, f_3 \\ &+11\, yz\, f_3 + 22\, z^2\, f_3 - \boxed{x^2\, f_2} - 18\, xy\, f_2 - 19\, y^2\, f_2 \\ &-8\, xz\, f_2 - 5\, yz\, f_2 - 7\, z^2\, f_2 = 0 \end{aligned}$$

We can remove the row $x^2 f_2$

same way $f_1 f_3 - f_3 f_1 = 0 \longrightarrow$ remove $x^2 f_1$

but $f_1 f_2 - f_2 f_1 = 0 \longrightarrow$ remove $x^2 f_1$! ???

Combining trivial relations

$$
\begin{aligned}
0 &= (f_2 f_1 - f_1 f_2) - 3(f_3 f_1 - f_1 f_3) \\
0 &= (f_2 - 3f_3) f_1 - f_1 f_2 + 3 f_1 f_3 \\
0 &= \boxed{f_4} f_1 - f_1 f_2 + 3 f_1 f_3 \\
0 &= \boxed{\left((1 - b)xy + 4\,yz + 2\,xz + 3\,y^2 - z^2\right) f_1} \\
&\quad -(6x^2 + \cdots) f_2 + 3(6x^2 + \cdots) f_3
\end{aligned}
$$

- if $b \neq 1$ remove $x\,y\ f_1$
- if $b = 1$ remove $y\,z\ f_1$

Need “some” computation

Degree 4 I

$$y^2 f_1, x\, z f_1, y\, z f_1, z^2 f_1, x\, y f_2, y^2 f_2, x\, z f_2,$$
$$y\, z f_2, z^2 f_2, x^2 f_3, x\, y f_3, y^2 f_3, x\, z f_3, y\, z f_3, z^2 f_3$$

In order to use previous computations (degree 2 and 3):

$$x f_2 \to f_6 \quad f_2 \to f_4$$
$$x f_1 \to f_8 \quad y f_1 \to f_7$$
$$f_1 \to f_5$$

$$y f_7, z f_8, z f_7, z^2 f_5, y f_6, y^2 f_4, z f_6, y\, z f_4,$$
$$z^2 f_4, x^2 f_3, x\, y f_3, y^2 f_3, x\, z f_3, y\, z f_3, z^2 f_3,$$

Degree 4 II

$$A_4 := \begin{bmatrix}
1 & 18 & 19 & 0 & 0 & 8 & 5 & 0 & 0 & 7 & 0 & 0 & 0 & 0 & 0 \\
 & 1 & 18 & 19 & 0 & 0 & 8 & 5 & 0 & 0 & 7 & 0 & 0 & 0 & 0 \\
 & & 1 & 18 & 19 & 0 & 0 & 8 & 5 & 0 & 0 & 7 & 0 & 0 & 0 \\
 & & & 1 & 3 & 0 & 0 & 2 & 4 & 0 & 0 & 22 & 0 & 0 & 0 \\
 & & & & 1 & 0 & 0 & 0 & 8 & 0 & 1 & 18 & 0 & 15 & 0 \\
 & & & & & 1 & 18 & 19 & 0 & 8 & 5 & 0 & 7 & 0 & 0 \\
 & & & & & & 1 & 18 & 19 & 0 & 8 & 5 & 0 & 7 & 0 \\
 & & & & & & & 1 & 3 & 0 & 2 & 4 & 0 & 22 & 0 \\
 & & & & & & & & 1 & 0 & 0 & 8 & 1 & 18 & 15 \\
 & & & & & & & & & 1 & 18 & 19 & 8 & 5 & 7 \\
 & & & & & & & & & & 1 & 11 & 0 & 13 & 0 \\
 & & & & & & & & & & & 1 & 12 & 20 & 18 \\
 & & & & & & & & & & & & 1 & 11 & 13 \\
 & & & & & & & & & & & & & 1 & 18 \\
 & & & & & & & & & & 1 & 3 & 2 & 4 & 22
\end{bmatrix}$$

Degree 4 III

$$A_4 := \left[\begin{array}{cccccccccc|ccccc}
1 & 18 & 19 & 0 & 0 & 8 & 5 & 0 & 0 & 7 & 0 & 0 & 0 & 0 & 0 \\
 & 1 & 18 & 19 & 0 & 0 & 8 & 5 & 0 & 0 & 7 & 0 & 0 & 0 & 0 \\
 & & 1 & 18 & 19 & 0 & 0 & 8 & 5 & 0 & 0 & 7 & 0 & 0 & 0 \\
 & & & 1 & 3 & 0 & 0 & 2 & 4 & 0 & 0 & 22 & 0 & 0 & 0 \\
 & & & & 1 & 0 & 0 & 0 & 8 & 0 & 1 & 18 & 0 & 15 & 0 \\
 & & & & & 1 & 18 & 19 & 0 & 8 & 5 & 0 & 7 & 0 & 0 \\
 & & & & & & 1 & 18 & 19 & 0 & 8 & 5 & 0 & 7 & 0 \\
 & & & & & & & 1 & 3 & 0 & 2 & 4 & 0 & 22 & 0 \\
 & & & & & & & & 1 & 0 & 0 & 8 & 1 & 18 & 15 \\
 & & & & & & & & & 1 & 18 & 19 & 8 & 5 & 7 \\
\hline
 & & & & & & & & & & 1 & 11 & 0 & 13 & 0 \\
 & & & & & & & & & & & 1 & 12 & 20 & 18 \\
 & & & & & & & & & & & & 1 & 11 & 13 \\
 & & & & & & & & & & & & & 1 & 18 \\
 & & & & & & & & & & 1 & 3 & 2 & 4 & 22
\end{array}\right]$$

Degree 4 IV

We need to consider only a small sub-matrix:

$$A_4' := \begin{array}{c} \\ yf_7 \\ z^2f_5 \\ zf_7 \\ zf_8 \\ z^2f_4 \end{array} \begin{array}{c} \begin{array}{ccccc} xyz^2 & y^2z^2 & xz^3 & yz^3 & z^4 \end{array} \\ \begin{pmatrix} 1 & 11 & 0 & 13 & 0 \\ & 1 & 12 & 20 & 18 \\ & & 1 & 11 & 13 \\ & & & 1 & 18 \\ 1 & 3 & 2 & 4 & 22 \end{pmatrix} \end{array}$$

F5 Criterion : analysis

Example: compute a Gröbner basis of $[f_1, f_2, f_3]$

Any combination of the trivial relations $f_i f_j = f_j f_i$ can always be written:

$$u(f_2 f_3 - f_3 f_2) + v(f_1 f_3 - f_3 f_1) + w(f_2 f_1 - f_1 f_2) = 0$$

F5 Criterion : analysis

Example: compute a Gröbner basis of $[f_1, f_2, f_3]$

Any combination of the trivial relations $f_i f_j = f_j f_i$ can always be written:

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where u, v, w are arbitrary polynomials.

$$\boxed{(w\, f_2 - v\, f_3)}\, f_1 + u\, f_2 f_3 - u\, f_3 f_2 + v\, f_1 f_3 - w f_1 f_2 = 0$$

$$\boxed{(w\, f_2 - v\, f_3)}\, f_1 \longrightarrow 0$$

F5 Criterion : analysis

Example: compute a Gröbner basis of $[f_1, f_2, f_3]$

Any combination of the trivial relations $f_i f_j = f_j f_i$ can always be written:

$$u(f_2 f_3 - f_3 f_2) + v(f_1 f_3 - f_3 f_1) + w(f_2 f_1 - f_1 f_2) = 0$$

where u, v, w are arbitrary polynomials.

$$\boxed{(w\, f_2 - v\, f_3)}\, f_1 + u\, f_2 f_3 - u\, f_3 f_2 + v\, f_1 f_3 - w f_1 f_2 = 0$$

$$\boxed{(w\, f_2 - v\, f_3)}\, f_1 \longrightarrow 0$$

(trivial) relation $h\, f_1 + \cdots = 0 \leftrightarrow \; h \in \mathrm{Id}(f_2, f_3)$

F5 Criterion : analysis

Example: compute a Gröbner basis of $[f_1, f_2, f_3]$

Any combination of the trivial relations $f_i f_j = f_j f_i$ can always be written:

$$u(f_2 f_3 - f_3 f_2) + v(f_1 f_3 - f_3 f_1) + w(f_2 f_1 - f_1 f_2) = 0$$

where u, v, w are arbitrary polynomials.

$$\boxed{(w\, f_2 - v\, f_3)}\, f_1 + u\, f_2 f_3 - u\, f_3 f_2 + v\, f_1 f_3 - w f_1 f_2 = 0$$

$$\boxed{(w\, f_2 - v\, f_3)}\, f_1 \longrightarrow 0$$

(trivial) relation $h\, f_1 + \cdots = 0 \leftrightarrow \; h \in \mathrm{Id}(f_2, f_3)$

F_5 Criterion: compute a Gröbner basis G' of $Id(f_2, f_3)$.

Remove row $t\, f_1$ iff t reducible by $\mathrm{LT}(G')$

Keep row $t\, f_1$ iff t not reducible by $\mathrm{LT}(G')$

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